ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{a^2-ab+b^2}{b^2-bc+c^2}+\frac{R^3}{r^3}\ge 8+\sum \frac{2b^2-3bc+2c^2}{2a^2-3ab+2b^2}$$

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Solution by Tapas Das-India

$$\sum \frac{a^2-ab+b^2}{b^2-bc+c^2}\overset{AM-GM}{\ge} 3\ (1)$$

$$\sum \frac{2b^2-3bc+2c^2}{2a^2-3ab+2b^2}\overset{AM-GM}{\le}\sum \frac{2b^2-3bc+2c^2}{4ab-3ab}=\sum\left(\frac{2b}{a}-\frac{3c}{a}+\frac{2c^2}{ab}\right)=$$

$$=2\sum\frac{b}{a}-3\sum\frac{c}{a}+\frac{2}{abc}\sum c^3=2\sum\frac{b}{a}-3\sum\frac{c}{a}+\frac{2}{abc}2s(s^2-3r^2-6Rr)\le$$

$$\overset{CBS\ \&\ AM-GM}{\le} 2\sqrt{\sum b^2\ \sum\frac{1}{a^2}}-3\cdot 3+\frac{s^2-3r^2-6Rr}{Rr}$$

$$\overset{Steinig\ \&\ Leibniz\ \&\ Gerretsen}{\le} 2\sqrt{9R^2\frac{1}{4r^2}}-9+\frac{4R^2-2Rr}{Rr}=\frac{7R}{r}-11$$

$$We\ need\ to\ show:$$

$$3+\frac{R^3}{r^3}\ge\frac{7R}{r}-11\ or,x^3-7x+6\overset{\frac{R}{r}=x\ge 2\ Euler}{\ge}0$$

$$(x-2)(x^2+2x-3)\ge 0\ true$$

Equality holds for an equilateral triangle.