ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$2024\sum\frac{m_a^2}{h_b^2+h_c^2}+2026\sum\frac{h_a^2}{m_b^2+m_c^2}+\frac{R^{2025}}{r^{2025}}\ge 2^{2025}+4050\sum\frac{a^4}{b^4+c^4}$$

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$$\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)=\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\overset{Bandila}{\le}\left(\frac{R}{r}\right)^2-2$$

$$\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)^2=\left(\left(\frac{R}{r}\right)^2-2\right)^2=\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+4$$

$$\sum\frac{a^4}{b^4+c^4}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(\frac{a^4}{b^4}+\frac{a^4}{c^4}\right)=\frac{1}{4}\sum\left(\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)^2-2\right)=$$

$$=\frac{1}{4}\sum\left(\left(\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+4\right)-2\right)=\frac{3}{4}\left(\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+2\right)$$

$$\sum\frac{m_a^2}{h_b^2+h_c^2}\overset{m_a\ge h_a}{\ge}\sum\frac{m_a^2}{m_b^2+m_c^2}\overset{Nesbitt}{\ge}\frac{3}{2}\ and$$

$$\sum\frac{h_a^2}{m_b^2+m_c^2}\overset{panatipal}{\ge}\frac{4r^2}{R^2}\sum\frac{h_a^2}{h_b^2+h_c^2}\overset{Nesbitt}{\ge}\frac{4r^2}{R^2}.\frac{3}{2}$$

Now we need to show:

$$2024\sum\frac{m_a^2}{h_b^2+h_c^2}+2026\sum\frac{h_a^2}{m_b^2+m_c^2}+\frac{R^{2025}}{r^{2025}}\ge 2^{2025}+4050\sum\frac{a^4}{b^4+c^4}$$

$$or\ 2024.\frac{3}{2}+2026\frac{4r^2}{R^2}.\frac{3}{2}+\left(\frac{R}{r}\right)^{2025}\ge 2^{2025}+4050.\frac{3}{4}\left(\left(\frac{R}{r}\right)^4-4\left(\frac{R}{r}\right)^2+2\right)$$

$$3036+\frac{12156}{x^2}+x^{2025}\ge 2^{2025}+\frac{6075}{2}(x^4-4x^2+2)\quad(1)$$

$$\left(we\ take\ x=\frac{R}{r}\ge 2(Euler)\right)$$

$$2x^{2027}-6075x^6+24300x^4-2(3039+2^{2025})x^2+24312\ge 0$$

$$2x^2\left(x^{2025}-2^{2025}\right)-6075x^6+24300x^4-6078x^2+24312\geq 0$$

$$2x^2(x-2)(x^{2024}+2x^{2023}+2^2x^{2022}+\cdots+2^{2024})-6075x^6+24300x^4-6078x^2+24312\geq 0$$

$$(x-2)\left(2x^2(x^{2024}+2x^{2023}+2^2x^{2022}+\cdots+2^{2024})-(x+2)(6075x^4+6078)\right)\geq 0$$

$$Now: (x+2)(6075x^4+6078)\leq (x+2)(6075x^4+6075x^4), (as\ x\geq 2\ )$$

$$or\ (x+2)(6075x^4+6078)\leq 12150(x+2)x^4$$

$$\frac{2x^{2026}}{12150(x+2)x^4}=\frac{2}{12150}\cdot\frac{x^{2026}}{(x+2)x^4}\overset{x\geq 2}{\geq}\frac{2}{12150}\cdot\frac{x^{2026}}{2x.x^4}=\frac{x^{2021}}{12150}=$$

$$=\frac{x^{2006}.x^{15}}{12150}\overset{x\geq 2}{\geq}\frac{x^{2006}.2^{15}}{12150}=\frac{x^{2006}.32768}{12150}>1$$

$$2x^{2026}>12150(x+2)x^4>(x+2)(6075x^4+6078)$$

$$(x-2)\left(2x^2\left(x^{2024}+2x^{2023}+2^2x^{2022}+\cdots+2^{2024}\right)-(x+2)\left(6075x^4+6078\right)\right)\geq 0\ true$$

Equality holds for an equilateral triangle.