ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sec^2 A + \sec^2 B + \sec^2 C \geq \csc^2 \frac{A}{2} + \csc^2 \frac{B}{2} + \csc^2 \frac{C}{2}$$

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$$\sum_{cyc} \sec^2 A \overset{?}{\geq} \sum_{cyc} \csc^2 \frac{A}{2} \Leftrightarrow \sum_{cyc} \tan^2 A \overset{?}{\geq} \sum_{cyc} \cot^2 \frac{A}{2}$$

$$\Leftrightarrow \left(\prod_{vyc} \tan A\right)^2 - 2\left(\prod_{vyc} \tan A\right)\left(\sum_{cyc} \cot A\right) \overset{?}{\geq} \frac{s^2}{r^2 s^4}\left(s^4 - 2s^2 r(4R+r)\right)$$

$$\Leftrightarrow \left(\frac{2rs}{s^2 - 4R^2 - 4Rr - r^2}\right)^2 - 2\left(\frac{2rs}{s^2 - 4R^2 - 4Rr - r^2}\right)\left(\frac{2(s^2 - 4Rr - r^2)}{4rs}\right)$$

$$\overset{?}{\geq} \frac{s^2 - 8Rr - 2r^2}{r^2}$$

$$\Leftrightarrow -s^6 + (8R^2 + 16Rr + 2r^2)s^4 - (16R^4 + 96R^3 r + 96R^2 r^2 + 24Rr^3 - 3r^4)s^2 +$$

$$r\left(128R^5 + 288R^4 r + 224R^3 r^2 + 72R^2 r^3 + 8Rr^4\right) \boxed{\overset{?}{\underset{①}{\geq}}} 0 \text{ and } \because P =$$

$$-(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3)(s^2 - 4R^2 - 4Rr - r^2) \overset{\text{Double-Rouche}}{\geq} 0$$

$(\because \Delta$ ABC $\to$ *acute* $\Rightarrow s > 2R + r)$ $\therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①} \overset{?}{\geq} P \Leftrightarrow -(8R - 3r)s^4 + (64R^3 + 28R^2 r + 2r^3)s^2 -$$

$$\left(128R^5 + 160R^4 r + 80R^3 r^2 + 28R^2 r^3 + 8Rr^4 + r^5\right) \boxed{\overset{?}{\underset{②}{\geq}}} 0 \text{ and } \because Q =$$

$$-(8R - 3r)(s^2 - 4R^2 - 4Rr - 3r^2)(s^2 - 4R^2 - 4Rr - r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove ②, it suffices to prove : LHS of ②} \overset{?}{\geq} Q \Leftrightarrow$$

$$24R^4 + 40R^3 r + 2R^2 r^2 - 16Rr^3 - 5r^4 \boxed{\overset{?}{\underset{③}{\geq}}} (6R^2 + 4Rr - 7r^2)s^2 \text{ and now,}$$

$$\text{RHS of ③} \overset{\text{Rouche}}{\leq} (6R^2 + 4Rr - 7r^2)\left(2R^2 + 10Rr - r^2 + 2(R - 2r).\sqrt{R^2 - 2Rr}\right)$$

$$\overset{?}{\leq} \text{LHS of ③} \Leftrightarrow 2(R - 2r)(6R^3 - 2R^2 r - 13Rr^2 + 3r^3) \overset{?}{\underset{(*)}{\geq}}$$

$$2(R - 2r).\sqrt{R^2 - 2Rr}.(6R^2 + 4Rr - 7r^2) \text{ and } \because R - 2r \overset{\text{Euler}}{\geq} 0 \text{ and}$$

$$6R^3 - 2R^2 r - 13Rr^2 + 3r^3 = (R - 2r)(6R^2 + 10Rr + 7r^2) + 17r^3 \overset{\text{Euler}}{\geq} 17r^3 > 0$$

$$\therefore \text{in order to prove } (*), \text{ it suffices to prove :}$$

$$(6R^3 - 2R^2 r - 13Rr^2 + 3r^3)^2 \overset{?}{>} (R^2 - 2Rr)(6R^2 + 4Rr - 7r^2)^2$$

$$\Leftrightarrow r^2(12R^4 + 6R^3 r + 2R^2 r(R - 2r) + 20Rr^3 + 9r^4) \overset{?}{>} 0 \to \text{true} \Rightarrow (*) \Rightarrow ③ \Rightarrow ②$$

$$\Rightarrow \textcircled{1} \text{ is true} \therefore \sec^2 A + \sec^2 B + \sec^2 C \geq \csc^2 \frac{A}{2} + \csc^2 \frac{B}{2} + \csc^2 \frac{C}{2}$$

$\forall$ acute Δ ABC, $'' ='' $ iff Δ ABC is equilateral (QED)