ROMANIAN MATHEMATICAL MAGAZINE

If $I - incenter\ in\ \Delta ABC$ the following relationship holds:

$$(AI+BI+CI)^2 \leq ab+bc+ca$$

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Solution by Tapas Das-India

$$AI+BI+CI=\frac{r}{\sin\frac{A}{2}}+\frac{r}{\sin\frac{B}{2}}+\frac{r}{\sin\frac{C}{2}}=r\sum\frac{1}{\sin\frac{A}{2}}=r\sum\sqrt{\frac{bc}{(s-b)(s-c)}}\overset{CBS}{\leq}$$

$$\leq r\sqrt{(ab+bc+ca)\left(\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}\right)}=$$

$$=r\sqrt{(ab+bc+ca)\left(\frac{s}{sr^2}\right)}=\sqrt{ab+bc+ca}$$

$$Therefore:$$

$$(AI+BI+CI)^2\leq ab+bc+ca$$

Equality holds for an equilateral triangle.