ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC the following relationship holds :**

$$\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\geq\frac{\sqrt{3}}{2}(a+b+c)$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\frac{m_a^2}{r_a}=\frac{1}{rs}\left(s\sum_{cyc}m_a^2-\sum_{cyc}\left(\frac{a}{4}\left(2\sum_{cyc}a^2-3a^2\right)\right)\right)=$$

$$=\frac{1}{rs}\left(\frac{3s(s^2-4Rr-r^2)}{2}-2s(s^2-4Rr-r^2)+\frac{3s}{2}(s^2-6Rr-3r^2)\right)=$$

$$=\frac{s^2-7Rr-4r^2}{r}\overset{?}{\geq}\frac{\sqrt{3}}{2}(a+b+c)=\sqrt{3}s\Leftrightarrow(s^2-7Rr-4r^2)^2\overset{?}{\geq}3s^2$$

$$\Leftrightarrow s^4-(14Rr+11r^2)s^2+r^2(49R^2+56Rr+16r^2)\underbrace{\overset{?}{\geq}}_{(*)}0\ \text{and}$$

$$\because(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore\ \text{in order to prove }(*),\text{it suffices to prove :}$$

$$\text{LHS of }(*)\overset{?}{\geq}(s^2-16Rr+5r^2)^2\Leftrightarrow(6R-7r)s^2\underbrace{\overset{?}{\geq}}_{(**)}r(69R^2-72Rr+3r^2)$$

$$\text{Now, }(6R-7r)s^2\overset{\text{Gerretsen}}{\geq}(6R-7r)(16Rr-5r^2)\overset{?}{\geq}r(69R^2-72Rr+3r^2)$$

$$\Leftrightarrow 3r^2(R-2r)(27R-16r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{m_a^2}{r_a}+\frac{m_b^2}{r_b}+\frac{m_c^2}{r_c}\geq\frac{\sqrt{3}}{2}(a+b+c)\ \forall\,\Delta\,ABC,''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$