ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$h_a^{h_b+h_c}.h_b^{h_a+h_c}.h_c^{h_a+h_b} \le \left(\frac{3R}{2}\right)^{9R}$$

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Solution by Tapas Das-India

$$\text{Known result} \sum h_a h_b = \frac{2rs^2}{R} \overset{Mitrinovic\ \&\ Euler}{\le} \frac{\frac{2R}{2}.\frac{27}{4}R^2}{R} = \frac{27R}{4}\ (1)$$

$$\sum h_a \overset{AM-HM}{\ge} \left(\frac{3}{\sum \frac{1}{h_a}}\right)^3 = (3r)^3 = 27r^3\ (2)$$

$$\sum h_a \overset{h_a \le m_a}{\le} \sum m_a \overset{Gotman}{\le} \frac{9R}{2}\ (3)$$

$$h_a^{h_b+h_c}.h_b^{h_a+h_c}.h_c^{h_a+h_b} \overset{AM-GM}{\le} \left(\frac{\sum h_c(h_a+h_b)}{\sum h_a+h_b}\right)^{\sum h_a+h_b} = \left(\frac{2\sum h_c h_b}{2\sum h_a}\right)^{2\sum h_a} \le$$

$$\le \left(\frac{\frac{27R}{4}}{\frac{9R}{2}}\right)^{\frac{2\times 9R}{2}} = \left(\frac{3R}{2}\right)^{9R}$$

Equality holds for an equilateral triangle.