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In any Δ ABC the following relationship holds :

$$(ab+bc+ca)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\geq 12$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right)\geq (s^2+4Rr+r^2)\left(\sum_{cyc}\frac{1}{r_ar_b}\right)$$

$$=\frac{(s^2+4Rr+r^2)(4R+r)}{s^2r}\overset{?}{\geq}12\Leftrightarrow(4R-11r)s^2+r(4R+r)^2\overset{?}{\geq}0\text{ and it's}$$

trivially true when : $4R-11r\geq 0$ and when : coefficient of $4R-11r<0$, then :

$$(4R-11r)s^2+r(4R+r)^2\overset{\text{Gerretsen}}{\geq}(4R-11r)(4R^2+4Rr+3r^2)+r(4R+r)^2$$

$$\overset{?}{\geq}0\Leftrightarrow 4R^3-3R^2r-6Rr^2-8r^3\overset{?}{\geq}0\Leftrightarrow(R-2r)(4R^2+5Rr+4r^2)\overset{?}{\geq}0\to\text{true}$$

$$\because R\overset{\text{Euler}}{\geq}2r\therefore(ab+bc+ca)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\geq 12\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$