ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$(m_a m_b + m_b m_c + m_c m_a)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right) \ge 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} m_a m_b\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right) \ge \left(\sum_{cyc} h_a h_b\right)\left(\frac{1}{r^2 s^4}\cdot\sum_{cyc} r_a^2 r_b^2\right)$$

$$= \left(\sum_{cyc}\frac{bc.\,ca}{4R^2}\right)\left(\frac{s^4 - 2s^2 r(4R+r)}{r^2 s^4}\right) = \frac{4Rrs.\,2s}{4R^2}\cdot\frac{s^2 - 8Rr - 2r^2}{r^2 s^2}$$

$$= \frac{2(s^2 - 8Rr - 2r^2)}{Rr} \overset{?}{\ge} 9 \Leftrightarrow 2s^2 \overset{?}{\ge} 25Rr + 4r^2 \to \text{true} \because 2s^2 \overset{\text{Gerretsen}}{\ge} 32Rr - 10r^2$$

$$= 25Rr + 4r^2 + 7r(R - 2r) \ge 25Rr + 4r^2 \left(\because R \overset{\text{Euler}}{\ge} 2r\right)$$

$$\therefore (m_a m_b + m_b m_c + m_c m_a)\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right) \ge 9 \ \forall \ \Delta \text{ ABC},$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$