ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a+b}{sin^2C}+\frac{b+c}{sin^2A}+\frac{c+a}{sin^2B}\ge 8\sqrt{3}R$$

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Solution by Daniel Sitaru-Romania

$$\frac{a+b}{sin^2C}+\frac{b+c}{sin^2A}+\frac{c+a}{sin^2B}=\sum_{cyc}\frac{a+b}{sin^2C}=\sum_{cyc}\frac{a+b}{\frac{c^2}{4R^2}}=4R^2\sum_{cyc}\frac{a+b}{c^2}\overset{AM-GM}{\ge}$$

$$\ge 4R^2\cdot 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{(abc)^2}}\overset{CESARO}{\ge}12R^2\sqrt[3]{\frac{8abc}{(abc)^2}}=$$

$$=\frac{24R^2}{\sqrt[3]{abc}}=\frac{24R^2}{\sqrt[3]{4Rrs}}\overset{EULER}{\ge}\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}s}}\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot\frac{3\sqrt{3}}{2}R}}=\frac{24R}{\sqrt[3]{\left(\sqrt{3}\right)^3}}=\frac{24R}{\sqrt{3}}=8\sqrt{3}R$$

Equality holds for $a=b=c$.