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In ΔABC the following relationship holds:

$$\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\right)^{\sin\frac{C}{2}}\left(1+\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\sin\frac{A}{2}}\left(1+\sin\frac{C}{2}\sin\frac{A}{2}\right)^{\sin\frac{B}{2}}\le\frac{5\sqrt{5}}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\sin\frac{A}{2}\overset{Jensen}{\le}3\sin\frac{A+B+C}{6}=3\sin\frac{\pi}{6}=\frac{3}{2}\ (1)$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\overset{AM-GM}{\le}\frac{\left(\sum\sin\frac{A}{2}\right)^3}{27}\ (2)$$

$$\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\right)^{\sin\frac{C}{2}}\left(1+\sin\frac{B}{2}\sin\frac{C}{2}\right)^{\sin\frac{A}{2}}\left(1+\sin\frac{C}{2}\sin\frac{A}{2}\right)^{\sin\frac{B}{2}}\overset{AM-GM}{\le}$$

$$\le\left(\frac{\sum\left(1+\sin\frac{B}{2}\sin\frac{C}{2}\right)\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}=\left(\frac{\sum\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\sum\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}\le$$

$$=\left(\frac{3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\sum\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}\overset{(2)}{\le}\left(\frac{3.\frac{\left(\sum\sin\frac{A}{2}\right)^3}{27}+\sum\sin\frac{A}{2}}{\sum\sin\frac{A}{2}}\right)^{\sum\sin\frac{A}{2}}=$$

$$=\left(\frac{1}{9}\left(\sum\sin\frac{A}{2}\right)^2+1\right)^{\sum\sin\frac{A}{2}}\overset{(1)}{\le}\left(\frac{1}{9}\left(\frac{3}{2}\right)^2+1\right)^{\frac{3}{2}}=\left(\frac{5}{4}\right)^{\frac{3}{2}}=\frac{5\sqrt{5}}{8}$$

Equality holds for an equilateral triangle.