ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$(\cos(A)\cos(B))^{\cos(C)}(\cos(C)\cos(B))^{\cos(A)}(\cos(A)\cos(C))^{\cos(B)}\le\frac{1}{8}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\prod_{cyc}(\cos(A)\cos(B))^{\cos(C)}\overset{Weighted\ (AM-GM)}{\le}\left(\frac{3.\cos(A).\cos(B).\cos(C)}{\cos(A)+\cos(B)+\cos(C)}\right)^{\sum_{cyc}\cos(A)}=$$

$$=\left(\frac{3.\frac{p^2-(2R+r)^2}{4R^2}}{1+\frac{r}{R}}\right)^{1+\frac{r}{R}}=\left(\frac{3(p^2-(2R+r)^2)}{4R^2\left(1+\frac{r}{R}\right)}\right)^{1+\frac{r}{R}}$$

$$(Here\ is\ an\ acute\ triangle. Therefore\ p>2R+r)$$

$$\overset{Gerretsen}{\le}\left(\frac{3(4R^2+4Rr+3r^2-4R^2-4Rr-r^2)}{4R(R+r)}\right)^{1+\frac{r}{R}}=\left(\frac{3r^2}{2R(R+r)}\right)^{1+\frac{r}{R}}$$

$$Let's\ prove\ that:\ \frac{3r^2}{2R(R+r)}\le\frac{1}{4}\to\{2R^2+2Rr-12r^2\ge 0\to$$

$$R^2+Rr-6r^2\ge 0\ \to\ (R-2r)(R+3r)\ge 0\to$$

$$R-2r\ge 0\ \to\ R\ge 2r\ (Euler)$$

$$Then:$$

$$\prod_{cyc}(\cos(A)\cos(B))^{\cos(C)}\le\left(\frac{1}{4}\right)^{1+\frac{r}{R}}\overset{Euler}{\le}\left(\frac{1}{4}\right)^{\frac{3}{2}}=\frac{1}{8}$$

$$Equality\ holds\ if: A=B=C.$$