ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{\cos\frac{B-C}{2}}{\sin^3\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin^3\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin^3\frac{C}{2}}\ge\frac{3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cos\frac{B-C}{2}}{\sin^3\frac{A}{2}}=\sum_{cyc}\frac{\frac{b+c}{a}}{\sin^2\frac{A}{2}}=\frac{1}{r^2}\sum_{cyc}\left(\frac{b+c}{a}.AI^2\right)$$

$$=\frac{1}{r^2}.\sum_{cyc}\left(\frac{bc(b+c)}{4Rrs}.(bc-4Rr)\right)$$

$$=\frac{1}{4Rr^3s}.\left(\sum_{cyc}\left(b^2c^2(2s-a)\right)-4Rr\sum_{cyc}\left(bc(2s-a)\right)\right)$$

$$=\frac{2s\sum_{cyc}a^2b^2-4Rrs\sum_{cyc}ab-8Rrs\sum_{cyc}ab+48R^2r^2s}{4Rr^3s}\overset{?}{\ge}\frac{3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{12R}{r}$$

$$\Leftrightarrow\sum_{cyc}a^2b^2-6Rr\sum_{cyc}ab\underbrace{\overset{?}{\ge}}_{(*)}0\ and\ indeed,\sum_{cyc}a^2b^2\ge$$

$$\frac{1}{3}\left(\sum_{cyc}ab\right)(s^2-14Rr+r^2+18Rr)$$

$$=\frac{1}{3}\left(\sum_{cyc}ab\right)\left((s^2-16Rr+5r^2)+2r(R-2r)+18Rr\right)\ge\frac{18Rr}{3}\left(\sum_{cyc}ab\right)$$

$$\left(\because s^2-16Rr+5r^2\overset{Gerretsen}{\ge}0\ \&\ 2r(R-2r)\overset{Euler}{\ge}0\right)and\ so,\sum_{cyc}a^2b^2\ge 6Rr\sum_{cyc}ab$$

$$\Rightarrow(*)\text{ is true}\therefore\frac{\cos\frac{B-C}{2}}{\sin^3\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin^3\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin^3\frac{C}{2}}\ge\frac{3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\ \forall\Delta ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)