ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$2\left(\frac{1}{\sin B}+\frac{1}{\sin C}-\sqrt{3}\right)\geq \cot B+\cot C$$

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$$Let\ u=\tan\frac{B}{2}>0, v=\tan\frac{C}{2}>0\ then\ \sin B=\frac{2u}{1+u^2}, \cos B=\frac{1-u^2}{1+u^2}$$

$$\frac{2-\cos B}{\sin B}=\frac{2-\frac{1-u^2}{1+u^2}}{\frac{2u}{1+u^2}}=\frac{1+3u^2}{2u}=\frac{1}{2u}+\frac{3u}{2}\ similarly:$$

$$\frac{2-\cos C}{\sin C}=\frac{1}{2v}+\frac{3v}{2}$$

$$2\left(\frac{1}{\sin B}+\frac{1}{\sin C}-\sqrt{3}\right)-\cot B+\cot C=$$

$$=2\left(\frac{1}{\sin B}-\frac{1}{2}\cot B\right)+2\left(\frac{1}{\sin C}-\frac{1}{2}\cot C\right)-2\sqrt{3}=$$

$$=2\frac{2-\cos B}{2\sin B}+2\frac{2-\cos C}{2\sin C}-2\sqrt{3}=\frac{1}{2u}+\frac{3u}{2}+\frac{1}{2v}+\frac{3v}{2}-2\sqrt{3}\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{\frac{1}{2u}\cdot\frac{3u}{2}}+2\sqrt{\frac{1}{2v}\cdot\frac{3v}{2}}-2\sqrt{3}=\sqrt{3}+\sqrt{3}-2\sqrt{3}=0$$

$$So: 2\left(\frac{1}{\sin B}+\frac{1}{\sin C}-\sqrt{3}\right)\geq \cot B+\cot C$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}.$$