ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sin A\sin B\cos C+\sin B\sin C\cos A+\sin C\sin A\cos B\le\frac{9}{8}$$

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$$\sin^2 A+\sin^2 B-\sin^2 C=\frac{1}{2}(2\sin^2 A+2\sin^2 B-2\sin^2 C)=$$

$$=\frac{1}{2}(1-\cos 2A+1-\cos 2B-1+\cos 2C)=\frac{1}{2}(1-(\cos 2A+\cos 2B)-\cos 2C)=$$

$$=\frac{1}{2}(1-2\cos(A+B)\cos(A-B)+2\cos^2 C-1)\overset{A+B+C=\pi}{=}$$

$$=\frac{1}{2}(2\cos C\cos(A-B)+2\cos^2 C)\overset{A+B+C=\pi}{=}$$

$$=\cos C\,(\cos(A-B)-\cos(A+B))=2\cos C\sin B\sin A$$

$$\sin A\,.\sin B\,.\cos C=\frac{1}{2}(\sin^2 A+\sin^2 B-\sin^2 C)$$

$$\sin A\sin B\cos C+\sin B\sin C\cos A+\sin C\sin A\cos B=$$

$$=\sum\sin A\sin B\cos C=\frac{1}{2}\sum(\sin^2 A+\sin^2 B-\sin^2 C)=$$

$$=\frac{1}{2}\sum\sin^2 A=\frac{1}{8R^2}\sum a^2\overset{Leibniz}{\le}\frac{9R^2}{8R^2}=\frac{9}{8}$$

Equality holds for an equilateral triangle.