ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\le\frac{9}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\cos A-\cos B+\cos C=(\cos A+\cos C)-\cos C=$$

$$=2\cos\frac{A+C}{2}\cos\frac{A-C}{2}-\cos B\overset{A+B+C=\pi}{=}2\sin\frac{B}{2}\cos\frac{A-C}{2}-\left(1-2\sin^2\frac{B}{2}\right)=$$

$$=2\sin\frac{B}{2}\left(\cos\frac{A-C}{2}+\sin\frac{B}{2}\right)-1\overset{A+B+C=\pi}{=}$$

$$=2\sin\frac{B}{2}\left(\cos\frac{A-C}{2}+\cos\frac{A+C}{2}\right)-1=4\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}-1$$

$$\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}=\frac{1}{4}(\cos A-\cos B+\cos C+1)$$

$$\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}=\sum\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}=$$

$$=\frac{1}{4}\sum(\cos A-\cos B+\cos C+1)=\frac{1}{4}\sum\cos A+\frac{3}{4}=$$

$$=\frac{1}{4}\left(1+\frac{r}{R}\right)+\frac{3}{4}\overset{Euler}{\le}\frac{1}{4}\left(1+\frac{1}{2}\right)+\frac{3}{4}=\frac{9}{8}$$

Equality holds for an equilateral triangle.