ROMANIAN MATHEMATICAL MAGAZINE

Let ABC be a triangle with area F and area of the triangle with side lengths $\sqrt{a(b+c-a)}, \sqrt{b(a+c-b)}, \sqrt{c(a+b-c)}$ *be* F'

$$\text{then prove that } \frac{F}{F'} \ge \sqrt{\frac{6r}{R} - \frac{R}{r}}$$

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Solution by Tapas Das-India

$$Let\ a' = \sqrt{a(b+c-a)} = \sqrt{2a(s-a)}, b' = \sqrt{b(a+c-b)} =$$

$$= \sqrt{2b(s-b)}, c' = \sqrt{c(c+a-c)} = \sqrt{2c(s-c)}$$

$$We\ know\ that: (a'b'c')^2 \overset{Carlitz}{\ge} \left(\frac{4F'}{\sqrt{3}}\right)^3$$

$$or (F')^3 \le \frac{3\sqrt{3}}{64}(a'b'c')^2 = \frac{3\sqrt{3}}{64} 8abc(s-a)(s-b)(s-c) = \frac{3\sqrt{3}}{64} 8(abc)(sr^2)$$

$$or\ F' \le \frac{\sqrt{3}}{2}(abc)^{\frac{1}{3}}(sr^2)^{\frac{1}{3}}$$

$$\frac{F}{F'} \ge \frac{\frac{abc}{4R}}{\frac{\sqrt{3}}{2}(abc)^{\frac{1}{3}}(sr^2)^{\frac{1}{3}}} = \frac{(abc)^{\frac{2}{3}}}{2R} \cdot \frac{1}{\sqrt{3}(sr^2)^{\frac{1}{3}}} = \frac{(4Rrs)^{\frac{2}{3}}}{2R} \cdot \frac{1}{\sqrt{3}(sr^2)^{\frac{1}{3}}} = \frac{2^{\frac{1}{3}}}{R^{\frac{1}{3}}} \cdot \frac{s^{\frac{1}{3}}}{\sqrt{3}} \ge$$

$$\overset{Mitrinovic}{\ge} \frac{2^{\frac{1}{3}}}{R^{\frac{1}{3}}} \cdot \frac{(3\sqrt{3}r)^{\frac{1}{3}}}{\sqrt{3}} = \left(\frac{2r}{R}\right)^{\frac{1}{3}} = \left(\frac{2r}{R}\frac{r^2}{r^2}\right)^{\frac{1}{3}} \overset{Euler}{\ge} \left(\frac{8r}{R}\frac{r^2}{R^2}\right)^{\frac{1}{3}} = \frac{2r}{R}$$

$$We\ need\ to\ show:$$

$$\frac{2r}{R} \ge \sqrt{\frac{6r}{R} - \frac{R}{r}}\ or\ \frac{4r^2}{R^2} \ge \frac{6r}{R} - \frac{R}{r}\ or\ R^3 - 6Rr^2 + 4r^3 \ge 0$$

$$or\ (R-2r)(R^2+2Rr-2r^2) \ge 0\ true\ by\ Euler$$

Equality holds for an equilateral triangle.