ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$18r^2 \le \sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-c)(s-b)} + \sqrt{ca(s-c)(s-a)} \le 2r(r_a + r_b + r_c)$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$Let\ x = a(s-a), y = b(s-b), z = c(s-c)$$
$$then: x + y + z = \sum a(s-a) = 2s^2 - 2(s^2 - r^2 - 4Rr) =$$
$$= 2r(4R + r) = 2r(r_a + r_b + r_c)$$

$$We\ \ need\ \ to\ show:$$
$$\sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-c)(s-b)} + \sqrt{ca(s-c)(s-a)} \le 2r(r_a + r_b + r_c)$$
$$or, \sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le x + y + z\ or, x + y + z - \sqrt{xy} - \sqrt{yz} - \sqrt{zx} \ge 0$$
$$or, \frac{1}{2}\left(\left(\sqrt{x} - \sqrt{y}\right)^2 + \left(\sqrt{y} - \sqrt{z}\right)^2 + \left(\sqrt{z} - \sqrt{x}\right)^2\right) \ge 0\ true$$

$$Now\ we\ \ need\ \ to\ show:$$
$$\sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-c)(s-b)} + \sqrt{ca(s-c)(s-a)} \ge 18r^2$$

$$\sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-c)(s-b)} + \sqrt{ca(s-c)(s-a)} \overset{AM-GM}{\ge}$$
$$\ge 3\sqrt[3]{abc(s-a)(s-b)(s-c)} = 3\sqrt[3]{4Rrs.sr^2} \overset{Euler\ \&\ Mitrinovic}{\ge} 3\sqrt[3]{8r^4.27r^2} = 18r^2$$

Equality holds for an equilateral triangle.