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In ΔABC the following relationship holds:

$$\sum\left(\frac{r_a}{r_b+r_c}\right)^n \geq \frac{3}{6^n}\left(\frac{4R+r}{s}\right)^{2n}, n \in N$$

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Solution by Tapas Das-India

$$\textit{Known result } \sum\frac{1}{s-a}=\frac{4R+r}{sr}$$

$$\frac{r_a}{r_b+r_c}=\frac{\frac{F}{s-a}}{\frac{F}{s-b}+\frac{F}{s-c}}=\frac{(s-b)(s-c)}{a(s-a)}=(s-a)(s-b)(s-c)\frac{\left(\frac{1}{s-a}\right)^2}{a}$$

$$\sum\frac{r_a}{r_b+r_c}=(s-a)(s-b)(s-c)\sum\frac{\left(\frac{1}{s-a}\right)^2}{a}\overset{Bergstrom}{\geq}$$

$$\geq (s-a)(s-b)(s-c)\frac{\left(\sum\frac{1}{s-a}\right)^2}{a+b+c}=$$

$$=\frac{(s-a)(s-b)(s-c)\left(\frac{r(4R+r)}{(s-a)(s-b)(s-c)}\right)^2}{a+b+c}=\frac{\left(r(4R+r)\right)^2}{2s.sr^2}=\frac{1}{2}\left(\frac{4R+r}{s}\right)^2$$

$$\sum\left(\frac{r_a}{r_b+r_c}\right)^n\overset{CBS}{\geq}\frac{1}{3^{n-1}}\left(\sum\frac{r_a}{r_b+r_c}\right)^n\geq\frac{1}{3^{n-1}}\left(\frac{1}{2}\left(\frac{4R+r}{s}\right)^2\right)^n=\frac{3}{6^n}\left(\frac{4R+r}{s}\right)^{2n}$$

Equality holds for an equilateral triangle.