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In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{r_a^3-9r^3}{r_a^2}\ge 6r$$

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$$\sum_{cyc}\frac{r_a^3-9r^3}{r_a^2}=\sum_{cyc}r_a-\frac{9r^3}{r^2s^2}.\sum_{cyc}(s-a)^2$$

$$=4R+r-\frac{9r}{s^2}.\left(3s^2-2s(2s)+2(s^2-4Rr-r^2)\right)$$

$$=\frac{(4R+r)s^2-9r(s^2-8Rr-2r^2)}{s^2}\overset{?}{\ge}6r\Leftrightarrow(2R-7r)s^2+9r^2(4R+r)\underbrace{\overset{?}{\ge}}_{(*)}0$$

and it's trivially true when : $2R-7r\ge 0$ and when : $2R-7r<0$, then :

$$\text{LHS of }(*)\overset{\text{Gerretsen}}{\ge}(2R-7r)(4R^2+4Rr+3r^2)+9r^2(4R+r)$$

$$=2(R-2r)(4R^2-2Rr+3r^2)\ge 0\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{r_a^3-9r^3}{r_a^2}\ge 6r\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$