ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a^3-9r^3}{h_a^2}\ge 6r$$

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$$\sum_{cyc}\frac{h_a^3-9r^3}{h_a^2}=\sum_{cyc}h_a-\frac{9r^3}{4r^2s^2}.\sum_{cyc}a^2=\frac{s^2+4Rr+r^2}{2R}-\frac{9r(s^2-4Rr-r^2)}{2s^2}$$

$$=\frac{s^2(s^2+4Rr+r^2)-9Rr(s^2-4Rr-r^2)}{2Rs^2}\overset{?}{\ge}6r$$

$$\Leftrightarrow s^4-(17Rr-r^2)s^2+9Rr^2(4R+r)\underbrace{\overset{?}{\ge}}_{(*)}0\ and\ \because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\ge}(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (15R-9r)s^2\underbrace{\overset{?}{\ge}}_{(**)}r(220R^2-169Rr+25r^2)\ and\ indeed,$$

$$(15R-9r)s^2-r(220R^2-169Rr+25r^2)\overset{\text{Gerretsen}}{\ge}$$

$$(15R-9r)(16Rr-5r^2)-r(220R^2-169Rr+25r^2)=10r(R-2r)(2R-r)$$

$$\overset{Euler}{\ge}0\Rightarrow(**)\Rightarrow(*)\text{ is true }\therefore\sum_{cyc}\frac{h_a^3-9r^3}{h_a^2}\ge 6r\ \forall\,\Delta\,ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)