ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{1}{3\cot^2 A + 1} \ge \frac{3}{2}$$

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$$\sum_{cyc} \frac{1}{3\tan^2 \frac{A}{2} + 1} = \sum_{cyc} \frac{s^2(3r_b^2+s^2)(3r_c^2+s^2)}{(3r_a^2+s^2)(3r_b^2+s^2)(3r_c^2+s^2)}$$

$$= \frac{9s^2\sum_{cyc} r_b^2r_c^2 + 6s^4\sum_{cyc} r_a^2 + 3s^6}{27r_a^2r_b^2r_c^2 + 9s^2\sum_{cyc} r_b^2r_c^2 + 3s^4\sum_{cyc} r_a^2 + s^6} \overset{?}{\ge} \frac{3}{2}$$

$$\Leftrightarrow 3s^4\sum_{cyc} r_a^2 + 3s^6 \overset{?}{\ge} 9s^2\sum_{cyc} r_b^2r_c^2 + 81r^2s^4$$

$$\Leftrightarrow s^2((4R+r)^2 - 2s^2) + s^4 \overset{?}{\ge} 3\left(s^4 - 2s^2r(4R+r)\right) + 27r^2s^2$$

$$\Leftrightarrow s^2 \overset{?}{\le} 4R^2 + 8Rr - 5r^2 \Leftrightarrow (s^2 - 4R^2 - 4Rr - 3r^2) - 4r(R-2r) \overset{?}{\le} 0 \to \text{true}$$

$$\because s^2 - 4R^2 - 4Rr - 3r^2 \overset{\text{Gerretsen}}{\le} 0 \text{ and } 4r(R-2r) \overset{\text{Euler}}{\ge} 0$$

$$\therefore \sum_{cyc} \frac{1}{3\tan^2 \frac{A}{2} + 1} \ge \frac{3}{2}\ \forall\, \Delta\, ABC, \text{and implementing it on a triangle with angles :}$$

$$(\pi - 2A), (\pi - 2B), (\pi - 2C), \text{we get : } \sum_{cyc} \frac{1}{3\tan^2 \frac{\pi - 2A}{2} + 1} \ge \frac{3}{2}$$

$$\Rightarrow \sum_{cyc} \frac{1}{3\cot^2 A + 1} \ge \frac{3}{2} \text{ and since the latter triangle is an acute triangle,}$$

$$\therefore \sum_{cyc} \frac{1}{3\cot^2 A + 1} \ge \frac{3}{2}\ \forall \text{ acute } \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$