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In any ΔABC the following relationship holds :

$$\sum_{\text{cyc}} \frac{1}{3 \tan^2 \frac{A}{2} + 1} \geq \frac{3}{2}$$

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$$\begin{aligned} \sum_{\text{cyc}} \frac{1}{3 \tan^2 \frac{A}{2} + 1} &= \sum_{\text{cyc}} \frac{s^2(3r_b^2 + s^2)(3r_c^2 + s^2)}{(3r_a^2 + s^2)(3r_b^2 + s^2)(3r_c^2 + s^2)} \\ &= \frac{9s^2 \sum_{\text{cyc}} r_b^2 r_c^2 + 6s^4 \sum_{\text{cyc}} r_a^2 + 3s^6}{27r_a^2 r_b^2 r_c^2 + 9s^2 \sum_{\text{cyc}} r_b^2 r_c^2 + 3s^4 \sum_{\text{cyc}} r_a^2 + s^6} \stackrel{?}{\geq} \frac{3}{2} \\ &\Leftrightarrow 3s^4 \sum_{\text{cyc}} r_a^2 + 3s^6 \stackrel{?}{\geq} 9s^2 \sum_{\text{cyc}} r_b^2 r_c^2 + 81r^2 s^4 \\ &\Leftrightarrow s^2((4R + r)^2 - 2s^2) + s^4 \stackrel{?}{\geq} 3(s^4 - 2s^2 r(4R + r)) + 27r^2 s^2 \\ &\Leftrightarrow s^2 \stackrel{?}{\leq} 4R^2 + 8Rr - 5r^2 \Leftrightarrow (s^2 - 4R^2 - 4Rr - 3r^2) - 4r(R - 2r) \stackrel{?}{\leq} 0 \rightarrow \text{true} \\ &\quad \because s^2 - 4R^2 - 4Rr - 3r^2 \stackrel{\text{Gerretsen}}{\leq} 0 \text{ and } 4r(R - 2r) \stackrel{\text{Euler}}{\geq} 0 \\ &\therefore \sum_{\text{cyc}} \frac{1}{3 \tan^2 \frac{A}{2} + 1} \geq \frac{3}{2} \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$