ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum\left(\frac{h_a}{h_b h_c}\right)^3 \geq \frac{1}{9r^3}$$

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Solution by Tapas Das-India

$$Let\ x=\frac{r}{h_a}, y=\frac{r}{h_b}, z=\frac{r}{h_c}\ then\ x+y+z=1\ as$$

$$\sum\frac{1}{h_a}=\frac{1}{r}, \frac{h_a}{h_b h_c}=\frac{\frac{1}{h_b h_c}}{\frac{1}{h_a}}=\frac{1}{r}\frac{\frac{r}{h_b}\cdot\frac{r}{h_c}}{\frac{r}{h_a}}=\frac{1}{r}\cdot\frac{yz}{x}$$

$$\sum\left(\frac{h_a}{h_b h_c}\right)^3=\frac{1}{r^3}\sum\left(\frac{yz}{x}\right)^3=\frac{1}{r^3}\frac{y^6z^6+z^6x^6+x^6y^6}{x^3y^3z^3}\overset{\forall a,b,c>0\ \sum a^2\geq\sum ab}{\geq}$$

$$\geq\frac{1}{r^3}\frac{x^3y^3z^3(x^3+y^3+z^3)}{x^3y^3z^3}\overset{CBS}{\geq}\frac{1}{r^3}\frac{1}{9}(x+y+z)^3=\frac{1}{9r^3}$$

Equality holds for an equilateral triangle.