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In ΔABC holds:

$$\frac{72r^2}{R} \le \sum_{cyc} \frac{h_a}{\sin^2\left(\frac{A}{2}\right)} \le \frac{8(R^4 - 7r^4)}{Rr^2}$$

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$$\sum_{cyc} \frac{h_a}{\sin^2\left(\frac{A}{2}\right)} \overset{Chebyshev}{\ge} \frac{1}{3}\sum_{cyc} h_a \cdot \sum_{cyc} \frac{1}{\sin^2\left(\frac{A}{2}\right)} =$$

$$\frac{1}{3}\sum_{cyc} h_a \cdot \sum_{cyc} \frac{2Rh_a}{r.r_a} = \frac{1}{3}\frac{p^2 + r^2 + 4Rr}{2R} \cdot \frac{2R}{r} \cdot \sum_{cyc} \frac{h_a}{r_a} \overset{Euler}{\ge}$$

$$\frac{1}{3} \cdot \frac{27r^2 + r^2 + 8r^2}{r} \cdot \sum_{cyc} \frac{h_a}{r_a} \overset{\substack{Chebyshev \\ Euler}}{\ge} \frac{1}{3} \cdot \frac{36r^2}{r} \cdot \frac{1}{3}\sum_{cyc} h_a \cdot \sum_{cyc} \frac{1}{r_a} =$$

$$4r.\frac{p^2 + r^2 + 4Rr}{2R} \cdot \frac{1}{r} \overset{Euler}{\ge} 2.\frac{36r^2}{R} = \frac{72r^2}{R}$$

Here the conditions of Chebyshev's theorem

are taken into account $: \ a \le b \le c \ ; \ \ h_a \ge h_b \ge h_c$

$$\sin\left(\frac{A}{2}\right) \le \sin\left(\frac{B}{2}\right) \le \sin\left(\frac{C}{2}\right) ; \frac{1}{r_a} \ge \frac{1}{r_b} \ge \frac{1}{r_c} ; \frac{1}{\sin\left(\frac{A}{2}\right)} \ge \frac{1}{\sin\left(\frac{B}{2}\right)} \ge \frac{1}{\sin\left(\frac{C}{2}\right)}$$

$$\sum_{cyc} \frac{h_a}{\sin^2\left(\frac{A}{2}\right)} = \sum_{cyc} \frac{h_a.bc}{(p-b)(p-c)} = \sum_{cyc} \frac{h_a.2F}{\sin(A).(p-b)(p-c)} =$$

$$\sum_{cyc} \frac{h_a.2F.2R}{a(p-b)(p-c)} = \sum_{cyc} \frac{h_a.2F.2R.h_a}{2F(p-b)(p-c)} = \sum_{cyc} \frac{2R.h_a^2}{(p-b)(p-c)} \le$$

$$2R\sum_{cyc} \frac{p(p-a)}{(p-b)(p-c)} = 2Rp\sum_{cyc} \frac{(p-a)}{(p-b)(p-c)} = 2Rp\frac{\sum_{cyc}(p-a)^2}{\prod_{cyc}(p-a)} =$$

$$2Rp\frac{(-p^2) + \sum_{cyc} a^2}{\prod_{cyc}(p-a)} = \frac{2Rp^2}{F^2}\left((-p^2) + \sum_{cyc} a^2\right) =$$

$$\frac{2Rp^2}{p^2r^2}\left((-p^2)+2p^2-2r^2-8Rr\right)=\frac{2R}{r^2}(p^2-2r^2-8Rr)\overset{Gerretsen}{\leq}$$

$$\frac{2R}{r^2}(4R^2+4Rr+3r^2-2r^2-8Rr)=\frac{2R}{r^2}(4R^2-4Rr+r^2)=$$

$$\frac{2R^2}{Rr^2}(4R^2-4Rr+r^2)=\frac{8R^4-8R^3r+2R^2r^2}{Rr^2}$$

Showing that the inequality is true

$$\frac{8R^4-8R^3r+2R^2r^2}{Rr^2}\leq\frac{8(R^4-7r^4)}{Rr^2}$$

$$8R^4-8R^3r+2R^2r^2\leq 8(R^4-7r^4)$$

$$8R^3r+2R^2r^2-56r^4\geq 0$$

$$4R^3r+R^2r^2-28r^4\geq 0$$

$$4\left(\frac{R}{r}\right)^3+\left(\frac{R}{r}\right)^2-28\geq 0$$

$$4\left(\left(\frac{R}{r}\right)^3-8\right)+\left(\frac{R}{r}\right)^2+4\geq 0$$

$$4\left(\frac{R}{r}-2\right)\left(\left(\frac{R}{r}\right)^2+2\left(\frac{R}{r}\right)+4\right)+\left(\frac{R}{r}\right)^2+4\geq 0$$

$$\frac{R}{r}-2\geq 0\ \rightarrow\ \frac{R}{r}\geq 2\ \rightarrow\ R\geq 2r\ (Euler)\ (true)$$

Equality holds for an equilateral triangle.