

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} \left(\sin^2(A) + \cos^2\left(\frac{A}{2}\right) \right) \leq \frac{17}{4} + \frac{r^2}{R^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\begin{aligned} \sum_{cyc} \left(\sin^2(A) + \cos^2\left(\frac{A}{2}\right) \right) &= \sum_{cyc} \left(\sin^2(A) \right) + \sum_{cyc} \left(\cos^2\left(\frac{A}{2}\right) \right) = \\ &= \frac{s^2 - r^2 - 4Rr}{2R^2} + \left(2 + \frac{r}{2R} \right) \stackrel{\text{Gerretsen}}{\leq} \\ &= \frac{4R^2 + 4Rr + 3r^2 - r^2 - 4Rr}{2R^2} + 2 + \frac{r}{2R} = \\ &= \frac{4R^2 + 4R^2 + 2r^2 + Rr}{2R^2} = \frac{8R^2 + 2r^2 + Rr}{2R^2} = \\ &= 4 + \frac{r}{2R} + \left(\frac{r}{R} \right)^2 \stackrel{\text{Euler}}{\leq} 4 + \frac{1}{4} + \left(\frac{r}{R} \right)^2 = \frac{17}{4} + \frac{r^2}{R^2} \end{aligned}$$

Equality holds if the triangle is an equilateral one.