ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\sqrt[3]{\frac{r_a}{3r}+\frac{21r}{r_b}}\le\frac{2p^2}{9r^2}$$

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$$\sum_{cyc}\sqrt[3]{\frac{r_a}{3r}+\frac{21r}{r_b}}-\frac{2p^2}{9r^2}=\frac{1}{4}.\sum_{cyc}\sqrt[3]{\left(\frac{1}{x}+7y\right).8.8}-2\sum_{cyc}\frac{1}{xy}$$

$$\left(x=\frac{3r}{r_a},y=\frac{3r}{r_b},y=\frac{3r}{r_c}\text{ and }\because\sum_{cyc}r_ar_b=p^2\right)\overset{AM-GM}{\le}$$

$$\frac{1}{12}\left(\sum_{cyc}\frac{1}{x}+7\sum_{cyc}x+3(8+8)\right)-2\sum_{cyc}\frac{1}{xy}$$

$$\le\frac{1}{12}\left(\frac{1}{3xyz}\left(\sum_{cyc}x\right)^2+7\sum_{cyc}x+48\right)-\frac{2}{xyz}\left(\sum_{cyc}x\right)=\frac{69}{12}-\frac{1}{xyz}\left(6-\frac{9}{36}\right)$$

$$\left(\because\sum_{cyc}x=3r.\sum_{cyc}\frac{1}{r_a}=3\right)=\frac{69}{12}-\frac{69}{12}.\frac{1}{xyz}\le\frac{69}{12}-\frac{69}{12}=0$$

$$\left(\because 3=\sum_{cyc}x\overset{AM-GM}{\ge}3.\sqrt{xyz}\Rightarrow xyz\le 1\right)\therefore\sum_{cyc}\sqrt[3]{\frac{r_a}{3r}+\frac{21r}{r_b}}\le\frac{2p^2}{9r^2}\ \forall\,\Delta\text{ ABC},$$

$$''='' \text{ iff } \Delta\text{ ABC is equilateral (QED)}$$