ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{h_a\sqrt{5+\frac{12r}{h_b}}}\geq\frac{1}{3r}$$

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Solution by Soumava Chakrabrty-Kolkata-India

$$\sum_{cyc}\frac{3r}{h_a.\sqrt{5+\frac{12r}{h_b}}}=\sum_{cyc}\frac{x}{\sqrt{5+4y}}\left(x=\frac{3r}{h_a},y=\frac{3r}{h_b},y=\frac{3r}{h_c}\right)=\sum_{cyc}\frac{x^{\frac{3}{2}}}{\sqrt{5x+4xy}}$$

$$\overset{Radon}{\geq}\frac{\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{5\sum_{cyc}x+4\sum_{cyc}xy}}\geq\frac{\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{5\sum_{cyc}x+\frac{4}{3}\left(\sum_{cyc}x\right)^2}}=\frac{\sqrt{27}}{\sqrt{15+12}}$$

$$\left(\because\sum_{cyc}x=3r.\sum_{cyc}\frac{1}{h_a}=3\right)\therefore\sum_{cyc}\frac{1}{h_a.\sqrt{5+\frac{12r}{h_b}}}\geq\frac{1}{3r}\ \forall\ \Delta\ ABC,$$

$$''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral (QED)}$$