ROMANIAN MATHEMATICAL MAGAZINE

In any acute $\Delta\, ABC$, the following relationship holds :

$$\sum_{cyc}\sqrt{\tan A+\cot A}\ge 2\sum_{cyc}\sqrt{\cot A}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\tan\frac{A}{2}+\cot\frac{A}{2}}=\sum_{cyc}\sqrt{\tan\frac{A}{2}+\frac{1}{\tan\frac{A}{2}}}=\sum_{cyc}\frac{\sec\frac{A}{2}}{\sqrt{\tan\frac{A}{2}}}\overset{AM-GM}{\ge}$$

$$3.\sqrt[3]{\frac{4R}{s}.\frac{1}{\sqrt{\frac{1}{s^3}.s^2r}}}\Rightarrow\sum_{cyc}\sqrt{\tan\frac{A}{2}+\cot\frac{A}{2}}\ge 3.\sqrt[3]{\frac{4R}{s}.\sqrt{\frac{s}{r}}}\to (1)\text{ and}$$

$$2\sum_{cyc}\sqrt{\tan\frac{A}{2}}\overset{CBS}{\le}2\sqrt{3}.\sqrt{\frac{4R+r}{s}}\overset{?}{\le}3.\sqrt[3]{\frac{4R}{s}.\sqrt{\frac{s}{r}}}\Leftrightarrow 729.\frac{16R^2}{s^2}.\frac{s}{r}\overset{?}{\ge}64.\frac{27(4R+r)^3}{s^3}$$

$$\Leftrightarrow 27R^2s^2\underbrace{\overset{?}{\ge}}_{(*)}4r(4R+r)^3;\text{ now, }27R^2s^2\overset{Gerretsen}{\ge}27R^2(16Rr-5r^2)\overset{?}{\ge}4r(4R+r)^3$$

$$\Leftrightarrow (R-2r)(176R^2+25Rr+2r^2)\overset{?}{\ge}0\to\text{true}\because R\overset{Euler}{\ge}2r\Rightarrow(*)\text{ is true}$$

$$\therefore 2\sum_{cyc}\sqrt{\tan\frac{A}{2}}\le 3.\sqrt[3]{\frac{4R}{s}.\sqrt{\frac{s}{r}}}\overset{via\ (1)}{\le}\sum_{cyc}\sqrt{\tan\frac{A}{2}+\cot\frac{A}{2}}$$

$$\therefore\sum_{cyc}\sqrt{\tan\frac{A}{2}+\cot\frac{A}{2}}\ge 2\sum_{cyc}\sqrt{\tan\frac{A}{2}}\text{ and implementing it on a triangle}$$

with angles $(\pi-2A),(\pi-2B),(\pi-2C)$, we get : $\sum_{cyc}\sqrt{\tan\frac{\pi-2A}{2}+\cot\frac{\pi-2A}{2}}$

$$\ge 2\sum_{cyc}\sqrt{\tan\frac{\pi-2A}{2}}\Rightarrow\sum_{cyc}\sqrt{\tan A+\cot A}\ge 2\sum_{cyc}\sqrt{\cot A}\text{ and since}$$

the latter triangle is an acute triangle, $\therefore\sum_{cyc}\sqrt{\tan A+\cot A}\ge 2\sum_{cyc}\sqrt{\cot A}$

$\forall$ acute $\Delta\, ABC$, " = " iff $\Delta\, ABC$ is equilateral (QED)