ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship ho$l$ds :

$$\sum_{\text{cyc}} \frac{\cot^2 A + \cot B \cot C}{\cot B + \cot C} \geq \sqrt{3}$$

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$$\forall\, x, y, z > 0, \sum_{\text{cyc}} \frac{x^2 + yz}{y+z} \overset{?}{\geq} \sum_{\text{cyc}} x$$

$$\Leftrightarrow \sum_{\text{cyc}} \left((x^2 + yz)\left(x^2 + \sum_{\text{cyc}} xy \right) \right) \overset{?}{\geq} \left(\sum_{\text{cyc}} x \right)\left(\left(\sum_{\text{cyc}} x \right)\left(\sum_{\text{cyc}} xy \right) - xyz \right)$$

$$\Leftrightarrow \sum_{\text{cyc}} x^4 + 2xyz\left(\sum_{\text{cyc}} x \right) + \left(\sum_{\text{cyc}} x^2 + \sum_{\text{cyc}} xy \right)\left(\sum_{\text{cyc}} xy \right) \overset{?}{\geq} \left(\sum_{\text{cyc}} xy \right)\left(\sum_{\text{cyc}} x \right)^2$$

$$\Leftrightarrow \sum_{\text{cyc}} x^4 \overset{?}{\geq} \sum_{\text{cyc}} x^2 y^2 \to \text{true} \therefore \sum_{\text{cyc}} \frac{x^2 + yz}{y+z} \geq \sum_{\text{cyc}} x \ \forall\, x, y, z > 0 \ and$$

$$\text{with } x \equiv \cot A,\ y \equiv \cot B,\ z \equiv \cot C, \text{ we get : } \sum_{\text{cyc}} \frac{\cot^2 A + \cot B \cot C}{\cot B + \cot C} \geq \sum_{\text{cyc}} \cot A$$

$$= \frac{\sum_{\text{cyc}} a^2}{4F} \overset{\text{Ionescu-Weitzenbock}}{\geq} \sqrt{3} \therefore \sum_{\text{cyc}} \frac{\cot^2 A + \cot B \cot C}{\cot B + \cot C} \geq \sqrt{3} \ \forall \text{ acute } \Delta\, ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)