ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{h_b h_c\left(\frac{r}{h_a}+1\right)}\le\frac{1}{4r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{h_b h_c\left(\frac{r}{h_a}+1\right)}=\sum_{cyc}\frac{bc}{4r^2s^2\left(\frac{ra}{2rs}+1\right)}=\sum_{cyc}\frac{bc}{\frac{4r^2s^2}{2s}(2s+a)}=$$

$$=\frac{2s}{4r^2s^2}\cdot\frac{1}{(2s+a)(2s+b)(2s+c)}\cdot\sum_{cyc}\left(bc(4s^2+2s(2s-a)+bc)\right)$$

$$=\frac{2s}{4r^2s^2}\cdot\frac{8s^2(s^2+4Rr+r^2)-24Rrs^2+(s^2+4Rr+r^2)^2-16Rrs^2}{2s(9s^2+6Rr+r^2)}\overset{?}{\le}\frac{1}{4r^2}$$

$$\Leftrightarrow(6R-9r)s^2\underbrace{\overset{?}{\ge}}_{(*)}r(4R+r)^2\ \textit{and indeed},(6R-9r)s^2\overset{\text{Gerretsen}}{\ge}$$

$$(6R-9r)(16Rr-5r^2)\overset{?}{\ge}r(4R+r)^2\Leftrightarrow 6r^2(R-2r)\ge 0\ \because R\overset{\text{Euler}}{\ge}2r\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{1}{h_b h_c\left(\frac{r}{h_a}+1\right)}\le\frac{1}{4r^2}\ \forall\,\Delta\,ABC,''='' \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$