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In any Δ ABC the following relationship ho$l$ds :

$$\sum_{cyc}\frac{1}{r_b r_c\left(\frac{r}{r_a}+1\right)}\le\frac{1}{4r^2}$$

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$$\sum_{cyc}\frac{1}{r_b r_c\left(\frac{r}{r_a}+1\right)}=\sum_{cyc}\frac{1}{s(s-a)\left(\frac{r(s-a)}{rs}+1\right)}=\frac{1}{s}\cdot\sum_{cyc}\frac{s}{(s-a)(2s-a)}$$

$$=\frac{1}{s}\cdot\sum_{cyc}\frac{(2s-a)-(s-a)}{(s-a)(2s-a)}=\frac{1}{s}\cdot\sum_{cyc}\frac{1}{s-a}-\frac{1}{s}\cdot\sum_{cyc}\frac{1}{b+c}$$

$$=\frac{1}{s}\cdot\frac{4R+r}{rs}-\frac{1}{2s^2(s^2+2Rr+r^2)}\cdot\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)$$

$$=\frac{4R+r}{s^2r}-\frac{4s^2+s^2+4Rr+r^2}{2s^2(s^2+2Rr+r^2)}\overset{?}{\le}\frac{1}{4r^2}\Leftrightarrow s^4-(14Rr-7r^2)s^2-2r^2(4R+r)^2\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(16Rr-5r^2)s^2-(14Rr-7r^2)s^2-2r^2(4R+r)^2=$$

$$(2Rr+2r^2)s^2-2r^2(4R+r)^2\overset{\text{Gerretsen}}{\ge}(2Rr+2r^2)(16Rr-5r^2)-2r^2(4R+r)^2$$

$$=6r^3(R-2r)\ge 0\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(*)\text{ is true }\therefore\sum_{cyc}\frac{1}{r_b r_c\left(\frac{r}{r_a}+1\right)}\le\frac{1}{4r^2}\ \forall\,\Delta\,ABC,$$

$''=''$ iff Δ ABC is equi$lateral$ (QED)