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In any ΔABC the following relationship holds :

$$\sum_{\text{cyc}} \sqrt{\tan \frac{A}{2} + \cot \frac{A}{2}} \geq 2 \sum_{\text{cyc}} \sqrt{\tan \frac{A}{2}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum_{\text{cyc}} \sqrt{\tan \frac{A}{2} + \cot \frac{A}{2}} &= \sum_{\text{cyc}} \sqrt{\tan \frac{A}{2} + \frac{1}{\tan \frac{A}{2}}} = \sum_{\text{cyc}} \frac{\sec \frac{A}{2}}{\sqrt{\tan \frac{A}{2}}} \stackrel{\text{AM-GM}}{\geq} \\ &3 \cdot \sqrt[3]{\frac{4R}{s} \cdot \frac{1}{\sqrt{\frac{1}{s^3} \cdot s^2 r}}} \Rightarrow \sum_{\text{cyc}} \sqrt{\tan \frac{A}{2} + \cot \frac{A}{2}} \geq 3 \cdot \sqrt[3]{\frac{4R}{s} \cdot \frac{\sqrt{s}}{r}} \rightarrow \textcircled{1} \text{ and} \\ 2 \sum_{\text{cyc}} \sqrt{\tan \frac{A}{2}} &\stackrel{\text{CBS}}{\leq} 2\sqrt{3} \cdot \sqrt{\frac{4R+r}{s}} \stackrel{?}{\leq} 3 \cdot \sqrt[3]{\frac{4R}{s} \cdot \frac{\sqrt{s}}{r}} \Leftrightarrow 729 \cdot \frac{16R^2}{s^2} \cdot \frac{s}{r} \stackrel{?}{\geq} 64 \cdot \frac{27(4R+r)^3}{s^3} \\ &\Leftrightarrow 27R^2 s^2 \stackrel{?}{\geq}_{(*)} 4r(4R+r)^3 \text{ and now, } 27R^2 s^2 \stackrel{\text{Gerretsen}}{\geq} 27R^2(16Rr - 5r^2) \\ &\stackrel{?}{\geq} 4r(4R+r)^3 \Leftrightarrow (R-2r)(176R^2 + 25Rr + 2r^2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \\ &\Rightarrow (*) \text{ is true} \therefore 2 \sum_{\text{cyc}} \sqrt{\tan \frac{A}{2}} \leq 3 \cdot \sqrt[3]{\frac{4R}{s} \cdot \frac{\sqrt{s}}{r}} \stackrel{\text{via } \textcircled{1}}{\leq} \sum_{\text{cyc}} \sqrt{\tan \frac{A}{2} + \cot \frac{A}{2}} \forall \Delta ABC, \\ &\quad \text{"=" iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$