ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC following relationship holds:

$$\sum \frac{1}{r_a\sqrt{5+\frac{12r}{r_b}}} \ge \frac{3}{r}$$

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Solution by Tapas Das-India

$$Let\, \frac{r}{r_a} = x, \frac{r}{r_b} = y, \frac{r}{r_c} = z \; then\; x+y+z = 1 \; (1)$$

$$\sum \frac{1}{r_a\sqrt{5+\frac{12r}{r_b}}} = \frac{1}{r}\sum \frac{\frac{r}{r_a}}{\sqrt{5+\frac{12r}{r_b}}} = \frac{1}{r}\sum \frac{x}{\sqrt{5+12y}} = \frac{1}{r}\sum \frac{x^2}{\sqrt{5x+12xy}} \overset{Bergstrom}{\ge}$$

$$\ge \frac{1}{r}\frac{(x+y+z)^2}{\sum\sqrt{5x+12xy}} \overset{CBS}{\ge} \frac{1}{r}\frac{(x+y+z)^2}{\sqrt{5(x+y+z)+12(xy+yz+zx)}} \ge$$

$$\ge \frac{1}{r}\frac{(x+y+z)^2}{\sqrt{5(x+y+z)+12\frac{(x+y+z)^2}{3}}} \overset{(1)}{=} \frac{1}{3r}$$

Equality holds for an equilateral triangle.