ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{r_a}{r_a^2 + 3rr_a + 9r^2} \le \frac{1}{3r}$$

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Solution by Tapas Das-India

$$Let\ x = \frac{r}{r_a}, y = \frac{r}{r_b}, z = \frac{r}{r_c}\ then\ x + y + z = 1$$

$$\sum \frac{r_a}{r_a^2 + 3rr_a + 9r^2} = \sum \frac{\frac{r_a}{r_a^2}}{\frac{r_a^2}{r_a^2} + \frac{3rr_a}{r_a^2} + \frac{9r^2}{r_a^2}} = \frac{1}{r} \sum \frac{\frac{r}{r_a}}{1 + 3\frac{r}{r_a} + 9\left(\frac{r}{r_a}\right)^2} =$$

$$= \frac{1}{r} \sum \frac{x}{1 + 3x + 9x^2} = \frac{1}{r} \sum \frac{x}{(1 + 9x^2) + 3x} \overset{AM-GM}{\le} \frac{1}{r} \sum \frac{x}{6x + 3x} = \frac{1}{3r}$$

Equality holds for an equilateral triangle.