In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{h_a^2h_b}\sum_{cyc}\frac{1}{h_b^2h_a}\ge\frac{64}{81R^6}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{1}{h_a^2h_b}\sum_{cyc}\frac{1}{h_b^2h_a}=\sum_{cyc}\frac{\frac{1}{h_a^2}}{h_b}\sum_{cyc}\frac{\frac{1}{h_b^2}}{h_a}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\sum_{cyc}h_b}\frac{\left(\sum_{cyc}\frac{1}{h_b}\right)^2}{\sum_{cyc}h_a}=\frac{\left(\frac{1}{r}\right)^4}{\frac{\left(s^2+r^2+4Rr\right)^2}{4R^2}}=$$

$$=\frac{4R^2}{r^4(s^2+r^2+4Rr)^2}\overset{Euler}{\geq}\frac{16}{R^4}\cdot\frac{4R^2}{(s^2+r^2+4Rr)^2}\overset{Gerretsen}{\geq}$$

$$\ge\frac{64}{R^2(4R^2+4Rr+3r^2+r^2+4Rr)^2}\overset{Euler}{\geq}\frac{64}{R^2(4R^2+4R^2+R^2)^2}=\frac{64}{81R^6}$$

Equality holds if the triangle is an equilateral one.

Solution 2 by Tapas Das-India

$$\sum\frac{1}{(h_ah_b)^{\frac{3}{2}}}=\sum\frac{1^{\frac{5}{2}}}{(h_ah_b)^{\frac{3}{2}}}\overset{Radon}{\geq}\frac{(1+1+1)^{\frac{5}{2}}}{(\sum h_ah_b)^{\frac{3}{2}}}=\frac{3^{\frac{5}{2}}}{\left(\frac{2rs^2}{R}\right)^{\frac{3}{2}}}\ (1)$$

$$\sum\frac{1}{h_a^2h_b}\sum\frac{1}{h_ah_b^2}\overset{C-S}{\geq}\left(\sum\sqrt{\frac{1}{h_a^2h_b}\cdot\frac{1}{h_ah_b^2}}\right)^2=$$

$$=\left(\sum\frac{1}{(h_ah_b)^{\frac{3}{2}}}\right)^2\overset{(1)}{\geq}\frac{3^5R^3}{2^3r^3s^6}\overset{Euler\ \&\ Mitrinovic}{\geq}\frac{3^5R^3}{\frac{2^3R^3}{2^3}\frac{3^9R^6}{2^6}}=\frac{64}{81R^6}$$

Equality holds for an equilateral triangle.