ROMANIAN MATHEMATICAL MAGAZINE

In *any* Δ ABC the fo*ll*owing re*la*tionship ho*l*ds :

$$\frac{9\sqrt{3}}{2}\left(\frac{r}{R}\right)^2 \le \sum_{cyc}\left((\sin^3 A)\cos(B-C)\right) \le \frac{9\sqrt{3}}{8}$$

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$$\sin 4A + \sin 4B + \sin 4C =$$
$$= 2\sin 2A\cos\left(2\left(\pi-(B+C)\right)\right) + 2\sin(2B+2C)\cos(2B-2C)$$
$$= 2\sin 2A\cos(2B+2C) + 2\sin(2\pi-2A)\cos(2B-2C)$$
$$= 2\sin 2A\cos(2B+2C) - 2\sin 2A\cos(2B-2C)$$
$$= 2(\sin 2A)(\cos(2B+2C)-\cos(2B-2C))$$
$$= -4\sin 2A\sin 2B\sin 2C \ \textit{and so,} \sum_{cyc}\left((\sin^3 A).\cos(B-C)\right)$$
$$= \frac{1}{2}\sum_{cyc}\left(\left(1-\frac{1+\cos 2A}{2}\right)(\sin 2B+\sin 2C)\right)$$
$$= \frac{1}{4}\sum_{cyc}(\sin 2B+\sin 2C) - \frac{1}{4}\sum_{cyc}\left((\cos 2A)\left(\sum_{cyc}\sin 2A - \sin 2A\right)\right)$$
$$= \frac{1}{2}.\frac{4rs}{2R^2} - \frac{1}{4}.\frac{4rs}{2R^2}.\left(-1-4\prod_{cyc}\cos A\right) + \frac{1}{8}.(-4\sin 2A\sin 2B\sin 2C)$$
$$= \frac{3rs}{2R^2} + \frac{2rs}{R^2}.\prod_{cyc}\cos A - \frac{1}{2}.8.\frac{4rs}{8R^2}.\prod_{cyc}\cos A \therefore \sum_{cyc}\left((\sin^3 A).\cos(B-C)\right) = \frac{3rs}{2R^2}$$

$$\textit{and} \because s \overset{\text{Mitrinovic}}{\ge} 3\sqrt{3}r \ \textit{and}\ rs \overset{\substack{\text{Euler} \\ \textit{and} \\ \text{Mitrinovic}}}{\le} \frac{R}{2}.\frac{3\sqrt{3}R}{2}$$

$$\therefore \frac{9\sqrt{3}}{2}\left(\frac{r}{R}\right)^2 \le \sum_{cyc}\left((\sin^3 A).\cos(B-C)\right) \le \frac{9\sqrt{3}}{8}\ \forall\,\Delta\,ABC,$$

$'' ='' $ iff Δ ABC is equi*lateral* (QED)