ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{4}{9}(R^2 - Rr - 2r^2) \le OI^2 + OG^2 \le \frac{2}{3}(3R^2 - 7Rr + 2r^2)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$OI^2 + OG^2 = R(R-2r) + \frac{9R^2 - 2(s^2 - 4Rr - r^2)}{9} \overset{?}{\le} \frac{2}{3}(3R^2 - 7Rr + 2r^2)$$

$$\Leftrightarrow 2(s^2 - 16Rr + 5r^2) \overset{?}{\ge} 0 \to \text{true via Gerretsen and } OI^2 + OG^2 =$$

$$R(R-2r) + \frac{9R^2 - 2(s^2 - 4Rr - r^2)}{9} \overset{?}{\ge} \frac{4}{9}(R^2 - Rr - 2r^2)$$

$$\Leftrightarrow 7R^2 - 3Rr + 5r^2 \overset{?}{\ge} s^2 \to \text{true} \because s^2 \overset{\text{Gerretsen}}{\le} 4R^2 + 4Rr + 3r^2$$

$$= 7R^2 - 3Rr + 5r^2 - (3R^2 - 7Rr + 2r^2) = 7R^2 - 3Rr + 5r^2 - (3R - r)(R - 2r)$$

$$\le 7R^2 - 3Rr + 5r^2 \left(\because R \overset{\text{Euler}}{\ge} 2r\right) \text{ and so, } \frac{4}{9}(R^2 - Rr - 2r^2) \le OI^2 + OG^2 \le$$

$$\frac{2}{3}(3R^2 - 7Rr + 2r^2)\ \forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$