ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{9r}{R^2 s} \le \sum \frac{\sin^3 B + \sin^3 C}{h_a^2} \le \frac{9R}{8r^2 s}$$

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$$\frac{\sin^3 B + \sin^3 C}{h_a^2} = \frac{a^2(b^3+c^3)}{8R^3 \times 4F^2} = \frac{1}{32R^3r^2s^2}\left(a^2(b^3+c^3)\right)$$

$$\sum a^3 . \sum a^2 = 2s(s^2-3r^2-6Rr) \times 2(s^2-r^2-4Rr) =$$

$$= 4s(s^4-4r^2s^2-10Rrs^2+3r^4+8Rr^3+24R^2r^2)$$

$$\sum a^5 = \left(\sum a\right)^5 - 5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca) =$$

$$= \left(\sum a\right)^5 - 5\big((a+b+c)(ab+bc+ca)-abc\big)(a^2+b^2+c^2+ab+bc+ca) =$$

$$= (2s)^5 - 5(2s(s^2+r^2+4Rr-4Rrs))\,(2(s^2-r^2-4Rr)+s^2+r^2+4Rr) =$$

$$= 2s(40R^2r^2+30Rr^3-10Rrs^2+5r^4-10r^2s^2+s^4)$$

$$\sum a^3 . \sum a^2 - \sum a^5 = 2s(s^2(s^2+2r^2-10Rr)+r^4+6Rr^3+8R^2r^2) \le$$

$$\overset{Gerretsen}{\le} 2s\Big((4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-10Rr)+r^4+6Rr^3 + 8R^2r^2\Big) = 2s(16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4)$$

$$\sum \frac{\sin^3 B + \sin^3 C}{h_a^2} = \frac{1}{32R^3r^2s^2}\sum\left(a^2(b^3+c^3)\right) =$$

$$= \frac{1}{32R^3r^2s^2}\left(\sum a^3 . \sum a^2 - \sum a^5\right) \le \frac{2s(16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4)}{32R^3r^2s^2}$$

We need to show:

$$\frac{2s(16R^4-8R^3r+16R^2r^2+8Rr^3+16r^4)}{32R^3r^2s^2} \le \frac{9R}{8r^2s}$$

$$-2R^3-8R^3r+16R^2r^2+8Rr^3+16r^4 \le 0$$

$$-2(R-2r)(R^3+6R^2r+4Rr^3+4r^3)\le 0 \ true\ by\ Euler.$$

$$\sum\frac{\sin^3 B+\sin^3 C}{h_a^2}=\frac{1}{2R}\sum\frac{b^3+c^3}{b^2c^2}\overset{AM-GM}{\geq}\frac{1}{R}\sum\frac{\sqrt{b^3c^3}}{(bc)^2}=$$

$$=\frac{1}{R}\sum\frac{1}{\sqrt{bc}}\overset{AM-GM}{\geq}\frac{1}{R}\sum\frac{2}{b+c}\overset{Bergstrom}{\geq}\frac{2}{R}\frac{9}{2(a+b+c)}=$$

$$=\frac{9}{R}\cdot\frac{1}{2s}=\frac{9r}{2Rrs}\overset{Euler}{\geq}\frac{9r}{\frac{2RR}{2}s}=\frac{9r}{R^2s}$$

Equality holds for an equilateral triangle.