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In ΔABC the following relationship holds:

$$\sum \frac{h_a}{\cos^2 \frac{A}{2}} \le 6R \le \sum \frac{r_a}{\cos^2 \frac{A}{2}}$$

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$$r_b + r_c = s\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = \frac{s \sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = s\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = 4Rcos^2\left(\frac{A}{2}\right) \ (1)$$

$$h_a \le w_a \le \sqrt{s(s-a)} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-a)(s-b)}} = \frac{F}{\sqrt{(s-b)(s-c)}} = \sqrt{r_b r_c} \ (2)$$

$$\sum \frac{h_a}{\cos^2\frac{A}{2}} = 4R\sum \frac{h_a}{r_b + r_c} \overset{(2)\&AM-GM}{\le} 4R\sum \frac{\sqrt{r_b r_c}}{2\sqrt{r_b r_c}} = 2R \times 3 = 6R \ (A)$$

$$\sum \frac{r_a}{\cos^2\frac{A}{2}} = 4R\sum \frac{r_a}{r_b + r_c} \overset{Nesbitt}{\ge} 4R \cdot \frac{3}{2} = 6R \ (B)$$

$$By\ (A)\ \&\ (B)\ we\ get \sum \frac{h_a}{\cos^2\frac{A}{2}} \le 6R \le \sum \frac{r_a}{\cos^2\frac{A}{2}}$$

Equality holds for an equilateral triangle.