ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$6R \leq \sum \frac{r_a}{\cos^2 \frac{A}{2}} \leq \frac{3R^3}{2r^2}$$

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$$r_b + r_c = s\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = \frac{s\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = s\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = 4R\cos^2\left(\frac{A}{2}\right)\ (1)$$

$$h_a \leq w_a \leq \sqrt{s(s-a)} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-a)(s-b)}} = \frac{F}{\sqrt{(s-b)(s-c)}} = \sqrt{r_b r_c}\ (2)$$

$$Known\ result: \sum \frac{a}{s-a} = \frac{4R-2r}{r}(3)$$

$$\sum \frac{r_a}{\cos^2\frac{A}{2}} = 4R\sum\frac{r_a}{r_b+r_c} \overset{AM-GM}{\leq} 4R\sum\frac{r_a}{2\sqrt{r_b r_c}} \overset{(2)}{\leq} 2R\sum\frac{r_a}{h_a} = R\sum\frac{a}{s-a}$$
$$= R\left(\frac{4R-2r}{r}\right) = \frac{R}{r^2}(4Rr - 2r^2)$$

$$We\ need\ to\ show:$$
$$\frac{R}{r^2}(4Rr-2r^2) \leq \frac{3R^3}{2r^2}\quad or\ 8Rr - 4r^2 \leq 3R^2$$
$$3R^2 - 8Rr + 4r^2 \geq 0\ \ or\ (R-2r)(3R-2r) \geq 0\ true\ by\ Euler.$$

$$\sum\frac{r_a}{\cos^2\frac{A}{2}} = 4R\sum\frac{r_a}{r_b+r_c} \overset{Nesbitt}{\geq} 4R\times\frac{3}{2} = 6R$$

Equality holds for an equilateral triangle.