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In $\Delta ABC$ the following relationship holds:

$$12r \le \sum \frac{h_a}{\cos^2\frac{A}{2}} \le 6R$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tapas Das-India*

$$r_b + r_c = s\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = \frac{s\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = s\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = 4R\cos^2\left(\frac{A}{2}\right) \ (1)$$

$$h_a \le w_a \le \sqrt{s(s-a)} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(s-a)(s-b)}} = \frac{F}{\sqrt{(s-b)(s-c)}} = \sqrt{r_b r_c} \ (2)$$

$$h_a.h_b.h_c \overset{GM-HM}{\ge} \left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^3 = 27r^3 \ (3)$$

$$\sum \frac{h_a}{\cos^2\frac{A}{2}} = 4R\sum\frac{h_a}{r_b+r_c} \overset{(2)\&AM-GM}{\le} 4R\sum\frac{\sqrt{r_b r_c}}{2\sqrt{r_b r_c}} = 2R\times 3 = 6R$$

$$\sum \frac{h_a}{\cos^2\frac{A}{2}} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{h_a.h_b.h_c}{\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}} \ge 3\sqrt[3]{27r^3.\frac{16R^2}{s^2}} \overset{Mitrinovic}{\ge}$$

$$\ge 3\sqrt[3]{27r^3.\frac{16.4s^2}{s^2 27}} = 12r$$

Equality holds for an equilateral triangle.