

ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{r_a \left(\frac{9r^2}{r_b^2} + 1 \right)} \geq \frac{1}{2r}$$

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Solution by Tapas Das-India

$$\text{Let } x = \frac{r}{r_a}, y = \frac{r}{r_b}, z = \frac{r}{r_c} \text{ then } x + y + z = r \sum \frac{1}{r_a} = \frac{r}{r} = 1 \quad (1)$$

$$\begin{aligned} \sum \frac{1}{r_a \left(\frac{9r^2}{r_b^2} + 1 \right)} &= \frac{1}{r} \sum \frac{\frac{r}{r_a}}{\left(\frac{9r^2}{r_b^2} + 1 \right)} = \frac{1}{r} \sum \frac{x}{9y^2 + 1} = \\ &= \frac{1}{r} \sum x \left(1 - \frac{9y^2}{1 + 9y^2} \right) \stackrel{AM-GM}{\geq} \frac{1}{r} \sum x \left(1 - \frac{9y^2}{6y} \right) = \frac{1}{r} \left(\sum x - \frac{3}{2} \sum xy \right) \geq \\ &\geq \frac{1}{r} \left(\sum x - \frac{(\sum x)^2}{3} \cdot \frac{3}{2} \right) \stackrel{(1)}{=} \frac{1}{2r} \end{aligned}$$

Equality holds for an equilateral triangle.