ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2Rr - r^2 \geq \frac{h_a h_b h_c}{h_a + h_b + h_c}$$

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$$2Rr - r^2 \overset{?}{\geq} \frac{h_a h_b h_c}{h_a + h_b + h_c} = \frac{2r^2s^2}{R.\frac{s^2+4Rr+r^2}{2R}}$$

$$\Leftrightarrow (2R - 5r)s^2 + r(8R^2 - 2Rr - r^2) \underbrace{\overset{?}{\geq}}_{(*)} 0$$

$$\text{Now, LHS of } (*) = (2R - 4r)s^2 - rs^2 + r(8R^2 - 2Rr - r^2) \overset{\text{Gerretsen}}{\geq}$$

$$(2R - 4r)(16Rr - 5r^2) - r(4R^2 + 4Rr + 3r^2) + r(8R^2 - 2Rr - r^2)$$

$$= 4r(R - 2r)(9R - 2r) \overset{\text{Euler}}{\geq} 0 \Rightarrow (*) \text{ is true} \therefore 2Rr - r^2 \geq \frac{h_a h_b h_c}{h_a + h_b + h_c}\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$