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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a}{\sin^2\frac{A}{2}}\ge\sum_{cyc}\frac{r_a}{\sin^2\frac{A}{2}}$$

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$$\sum_{cyc}\frac{h_a}{\sin^2\frac{A}{2}}=\sum_{cyc}h_a+s^2\sum_{cyc}\frac{h_a}{r_a^2}=$$

$$=\frac{s^2+4Rr+r^2}{2R}+\frac{s^2}{2Rr^2s^2}\cdot\sum_{cyc}\left(bc(s^2-2sa+a^2)\right)=$$

$$=\frac{s^2+4Rr+r^2}{2R}+\frac{1}{2Rr^2}\left(s^2(s^2+4Rr+r^2)-6sabc+abc(2s)\right)=$$

$$=\frac{s^2(s^2-12Rr+r^2)+r^2(s^2+4Rr+r^2)}{2Rr^2}\overset{?}{\ge}\sum_{cyc}\frac{r_a}{\sin^2\frac{A}{2}}=$$

$$=\frac{rs}{(s-a)(s-b)(s-c)}\cdot\sum_{cyc}bc=\frac{s^2+4Rr+r^2}{r}$$

$$\Leftrightarrow s^4-(14Rr-2r^2)s^2-r^2(8R^2-2Rr-r^2)\underbrace{\overset{?}{\ge}}_{(*)}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(2Rr-3r^2)s^2-r^2(8R^2-2Rr-r^2)\overset{\text{Gerretsen}}{\ge}$$

$$(2Rr-3r^2)(16Rr-5r^2)-r^2(8R^2-2Rr-r^2)=8r^2(R-2r)(3R-r)\overset{\text{Euler}}{\ge}0\Rightarrow$$

$$(*)\text{ is true }\therefore\sum_{cyc}\frac{h_a}{\sin^2\frac{A}{2}}\ge\sum_{cyc}\frac{r_a}{\sin^2\frac{A}{2}}\ \forall\,\Delta\,ABC,''=''\ \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$