ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{h_a\left(\frac{9r^2}{h_b^2}+1\right)} \geq \frac{1}{2r}$$

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Solution by Tapas Das-India

$$Let\ x=\frac{r}{h_a}, y=\frac{r}{h_b}, z=\frac{r}{h_c}\ then\ x+y+z=r\sum \frac{1}{h_a}=\frac{r}{r}=1\ (1)$$

$$\sum \frac{1}{h_a\left(\frac{9r^2}{h_b^2}+1\right)}=\frac{1}{r}\sum \frac{\frac{r}{h_a}}{\left(\frac{9r^2}{h_b^2}+1\right)}=\frac{1}{r}\sum \frac{x}{9y^2+1}=\frac{1}{r}\sum x\left(1-\frac{9y^2}{1+9y^2}\right)\overset{AM-GM}{\geq}$$

$$\geq \frac{1}{r}\sum x\left(1-\frac{9y^2}{6y}\right)=\frac{1}{r}\left(\sum x-\frac{3}{2}\sum xy\right)\geq \frac{1}{r}\left(\sum x-\frac{(\sum x)^2}{3}.\frac{3}{2}\right)\overset{(1)}{=}\frac{1}{2r}$$

Equality holds for an equilateral triangle.