ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a}{\sqrt{b+c-a}}+\frac{b}{\sqrt{a+c-b}}+\frac{c}{\sqrt{a+b-c}}\geq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a}{\sqrt{b+c-a}}+\frac{b}{\sqrt{a+c-b}}+\frac{c}{\sqrt{a+b-c}}\geq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

$$\sum_{cyc}\frac{a}{\sqrt{b+c-a}}=\sum_{cyc}\frac{a}{\sqrt{2}.\sqrt{p-a}}=\frac{1}{\sqrt{2}}\sum_{cyc}a\sqrt{\frac{r_a}{F}}=$$

$$=\frac{1}{\sqrt{2}}\sum_{cyc}a\sqrt{\frac{r_a}{F}}=\frac{1}{\sqrt{2F}}\sum_{cyc}a\sqrt{r_a}=\frac{1}{\sqrt{2F}}\sum_{cyc}\sqrt{a}\ .\ \sqrt{ar_a}$$

$$In\ \Delta\, ABC\ wlog\ a\leq b\leq c\ ,\qquad r_a\leq r_b\leq r_c$$

$$By\ Chebyshev's\ inequality\ :$$

$$\frac{1}{\sqrt{2F}}\sum_{cyc}\sqrt{a}\ .\ \sqrt{ar_a}\ \overset{Chebyshev}{\geq}$$

$$\frac{1}{3}\cdot\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{ar_a}+\sqrt{br_b}+\sqrt{cr_c}\right)\overset{AM-GM}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).3\sqrt[3]{\sqrt{(abc)(r_ar_br_c)}}=$$

$$=\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).\left(\sqrt[6]{(4RF).(rS^2)}\right)\overset{Euler}{\geq}\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).\left(\sqrt[6]{4.2r.F.(rS^2)}\right)=$$

$$=\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt[6]{8.r^2S^2.F}\right)=\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt[6]{8F^3}\right)=\sqrt{a}+\sqrt{b}+\sqrt{c}$$

$$Equality\ holds\ for:\ a=b=c$$