ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{r_a+r_b+r_c}{w_a+w_b+w_c} \ge \frac{p_a+p_b+p_c}{\sqrt{3}s} \ge \frac{h_a+h_b+h_c}{w_a+w_b+w_c}$$

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$$\sum_{cyc} p_a \le \frac{\sqrt{21s^2+4Rr+r^2}}{9s^2+6Rr+r^2}\sqrt{\begin{matrix}(s^2-16Rr-3r^2)(21s^2+4Rr+r^2)+\\4(8s^2(8Rr+2r^2)-4s^2(2Rr+2r^2)+4Rrs^2)\end{matrix}}$$

$$\left(\begin{matrix}\text{Solution to Inequality in Triangle by}\\\text{Dang Ngoc Minh} - 129;\text{www.ssmrmh.ro}\end{matrix}\right), \sum_{cyc} w_a \overset{\text{Bogdan Fustei}}{\le} \sqrt{\frac{(4R+r)(s^2+4Rr+r^2)}{2R}}$$

$$\therefore \text{to prove}: \frac{4R+r}{\sum_{cyc} w_a} \ge \frac{\sum_{cyc} p_a}{\sqrt{3}s}, \text{suffices}: 6s^2R(4R+r)(9s^2+6Rr+r^2)^2 \overset{?}{\ge}$$

$$(s^2+4Rr+r^2)(21s^2+4Rr+r^2)\left(\begin{matrix}(s^2-16Rr-3r^2)(21s^2+4Rr+r^2)+\\4(8s^2(8Rr+2r^2)-4s^2(2Rr+2r^2)+4Rrs^2)\end{matrix}\right)$$

$$\Leftrightarrow -441s^8+(1944R^2+570Rr+168r^2)s^6+$$
$$r(2592R^3+10184R^2r+5192Rr^2+702r^3)s^4+$$
$$X = r^2(864R^4+7608R^3r+5184R^2r^2+1218Rr^3+96r^4)s^2+$$
$$Y = r^4(1024R^4+960R^3r+336R^2r^2+52Rr^3+3r^4)\boxed{\underset{(*)}{\overset{?}{\ge}}}0 \text{ and } \because P =$$

$$-441s^4.(M = s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3) \overset{\text{Double-Rouche}}{\ge} 0$$
$$\therefore \text{to prove } (*),$$

$$\text{it suffices to prove}: \text{LHS of } (*) \overset{?}{\ge} P \Leftrightarrow (180R^2-8250Rr+1050r^2)s^6+$$

$$r(30816R^3+31352R^2r+10484Rr^2+1143r^3)s^4+X+Y\boxed{\underset{(**)}{\overset{?}{\ge}}}0 \to \text{true when}:$$

$$180R^2-8250Rr+1050r^2 \ge 0 \text{ and when}: 180R^2-8250Rr+1050r^2 < 0, Q =$$
$$(180R^2-8250Rr+1050r^2)s^2.M \overset{\text{Rouche}}{\ge} 0 \therefore \text{to prove } (**), \text{suffices}: \text{LHS}_{(**)} \overset{?}{\ge} Q$$
$$\Leftrightarrow (U = 720R^4+1416R^3r-16Rr^2(8113R-2999r)-957r^4)s^4-$$
$$r\left(\begin{matrix}32(360R^5-16257R^4r-10446R^3r^2-1675R^2r^3+98Rr^4+30r^5)+\\24R^3r^2-4R^2r^3-4Rr^4-6r^5\end{matrix}\right)s^2$$
$$+Y\boxed{\underset{(***)}{\overset{?}{\ge}}}0$$

$$\boxed{\text{Case 1}}\ U \ge 0 \text{ and then, since } T = U(s^2-16Rr+5r^2)^2+$$
$$4r\left(\begin{matrix}128\left(\begin{matrix}22R^5+1090R^4r-7488R^3r^2+\\5639R^2r^3-1003Rr^4+17r^5\end{matrix}\right)+\\64R^5+64R^4r+22R^3r^2+R^2r^3-15Rr^4-22r^5\end{matrix}\right)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\ge} 0$$
$$\therefore \text{to prove } (***), \text{it suffices to prove}: \text{LHS of } (***) \overset{?}{\ge} T$$

$$\Leftrightarrow (t-2)\begin{pmatrix}128(5617t^4-8747t^3+3520t^2-437t+6)+\\64t^4+68t^3+8t^2-47t+30\end{pmatrix}\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\rightarrow \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (***) \text{ is true } \boxed{Case\ 2}\ U<0 \text{ and then, since } T'=$$

$$U(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\overset{Double-Rouche}{\geq}0$$

$$\therefore \text{ in order to prove } (***), \text{it suffices to prove : LHS of } (***) \overset{?}{\geq} T' \Leftrightarrow$$

$$\begin{pmatrix}V=128(2R^6+8R^5r+27R^4r^2-2024R^3r^3+1239R^2r^4-115Rr^5+r^6)+\\104R^6+44R^5r+28R^4r^2-29R^3r^3+42R^2r^4-60Rr^5-8r^6\end{pmatrix}s^2$$

$$\boxed{\underset{①}{\overset{?}{\geq}}}\ r\begin{pmatrix}128\begin{pmatrix}45R^7+122R^6r-8038R^5r^2-3070R^4r^3+\\669R^3r^4+390R^2r^5+36Rr^6-r^7\end{pmatrix}+\\32R^6r-24R^5r^2+70R^4r^3-39R^3r^4+46R^2r^5-52Rr^6+r^7\end{pmatrix}$$

$$\boxed{Case\ 2a}\ V\geq 0; \text{LHS of ① } \overset{Rouche+Euler}{\geq} V\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right)\overset{?}{\geq} \text{RHS of ①}$$

$$\Leftrightarrow (t-2)\begin{pmatrix}128\begin{pmatrix}(t-2)(8437t^4-4126t^3+6404t^2+)\\(t-2)(9692t+19548)+39096\end{pmatrix}\\+(t-2)(64t^4+38t^3+51t^2+40t+60)\end{pmatrix}\overset{?}{\geq}0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$$\boxed{Case\ 2b}\ V<0 \text{ and then, LHS of ① } \overset{Gerretsen}{\geq} V(4R^2+4Rr+3r^2)\overset{?}{\geq} \text{RHS of ①} \Leftrightarrow$$

$$(t-2)\begin{pmatrix}128\left((t-2)\begin{pmatrix}5t^6+22t^5+81t^4+272t^3+\\771t^2+871t+1836\end{pmatrix}+3672\right)\\+(t-2)(80t^6+40t^5-4t^4+28t^3-31t^2+24t+60)\end{pmatrix}\overset{?}{\geq}0 \rightarrow \text{true}$$

$$\text{Again, }\left(\sum_{cyc} w_a\right)^2\cdot\left(\sum_{cyc} p_a\right)^2 \geq (s^2+12Rr+30r^2).\frac{100s^2-9r(64R-53r)}{25}\overset{?}{\geq}$$

$$\frac{3s^2(s^2+4Rr+r^2)^2}{4R^2}\begin{pmatrix}\text{Reference}-1: \text{Inequality in Triangle by Dang Ngoc Minh}-90;\\ \text{Reference}-2: \text{Inequality in Triangle by Amine Ben Ajiba}-65\end{pmatrix}$$

$$-75s^6+100(4R^2-6Rr-1{,}5r^2)s^4+r(2496R^3+12708R^2r-600Rr^2-75r^3)s^2$$

$$-r^2(27648R^4+46224R^3r-57240R^2r^2)\underset{(\bullet)}{\overset{?}{\geq}}0 \text{ and } \because \lambda =$$

$$(-75s^2-1900Rr)(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)+$$

$$4R(25R-50r)(s^2-16Rr+5r^2)^2\geq 0 \therefore \text{ to prove } (\bullet), \text{it suffices to prove :}$$

$$\text{LHS of } (\bullet)\overset{?}{\geq}\lambda \Leftrightarrow (724R^2-7273Rr+1525r^2)s^2+$$

$$Z=r(17088R^3+28044R^2r+11385Rr^2+1725r^3)\underset{(\bullet\bullet)}{\overset{?}{\geq}}0 \rightarrow \text{true when :}$$

$$724R^2-7273Rr+1525r^2\geq 0 \text{ and when it's} \leq 0, \text{LHS of } (\bullet\bullet)\overset{Gerretsen}{\geq}$$

$$(724R^2-7273Rr+1525r^2)(4R^2+4Rr+3r^2)+Z\overset{?}{\geq}0 \Leftrightarrow$$

$$(t-2)\left((t-2)(1448t^2+1238t+2772)+3969\right)\overset{?}{\geq}0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$$\therefore \frac{4R+r}{\sum_{cyc} w_a} \geq \frac{\sum_{cyc} p_a}{\sqrt{3}s} \geq \frac{\sum_{cyc} h_a}{\sum_{cyc} w_a}\ \forall\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$