ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$(w_a+w_b+w_c)(n_a+n_b+n_c)\ge(r_a+r_b+r_c)(h_a+h_b+h_c)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}w_a\right)^2\left(\sum_{cyc}n_a\right)^2\ge(s^2+12Rr+30r^2)(9s^2-80Rr-2r^2)$$

$$\left(\begin{array}{c}\text{Reference}-1:\text{Inequality in Triangle by Dang Ngoc Minh}-90;\\ \text{published at www.ssmrmh.ro}\\ \text{Reference}-2:\text{Inequality in Triangle by Mohamed Amine Ben Ajiba}-74;\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$\overset{?}{\ge}\left(\sum_{cyc}r_a\right)^2\left(\sum_{cyc}h_a\right)^2=(4R+r)^2.\frac{(s^2+4Rr+r^2)^2}{4R^2}$$

$$\Leftrightarrow(20R^2-8Rr-r^2)s^4-r(16R^3-976R^2r+24Rr^2+2r^3)s^2-$$

$$r^2(4096R^4+9952R^3r+336R^2r^2+16Rr^3+r^4)\boxed{\underset{(*)}{\overset{?}{\ge}}}0\text{ and }\because$$

$$(20R^2-8Rr-r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\therefore\text{ in order to prove }(*),$$

$$\text{it suffices to prove : LHS of }(*)\overset{?}{\ge}(20R^2-8Rr-r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(156R^3+130R^2r+6Rr^2+2r^3)s^2\boxed{\underset{(**)}{\overset{?}{\ge}}}r\left(\begin{array}{c}2304R^4+1176R^3r+\\465R^2r^2-6Rr^3-6r^4\end{array}\right)$$

$$\text{and again, LHS of }(**)\overset{\text{Gerretsen}}{\ge}(156R^3+130R^2r+6Rr^2+2r^3)(16Rr-5r^2)$$

$$\overset{?}{\ge}r(2304R^4+1176R^3r+465R^2r^2-6Rr^3-6r^4)$$

$$\Leftrightarrow192t^4+124t^3-1019t^2+8t-4\overset{?}{\ge}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(192t^3+508t^2-3t+2)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore(w_a+w_b+w_c)(n_a+n_b+n_c)\ge(r_a+r_b+r_c)(h_a+h_b+h_c)\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$