ROMANIAN MATHEMATICAL MAGAZINE

In *any* Δ ABC the *following* relationship holds :

$$\frac{m_a m_b m_c}{r_a r_b r_c} \geq \frac{p_a + p_b + p_c}{r_a + r_b + r_c} \geq \frac{h_a h_b h_c}{g_a g_b g_c}$$

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$$\sum_{\text{cyc}} p_a \leq \frac{\sqrt{21s^2+4Rr+r^2}}{9s^2+6Rr+r^2}\sqrt{\begin{matrix}(s^2-16Rr-3r^2)(21s^2+4Rr+r^2)+\\4(8s^2(8Rr+2r^2)-4s^2(2Rr+2r^2)+4Rrs^2)\end{matrix}}$$

$$\left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle by}\\ \text{Dang Ngoc Minh} - 129\text{; published at www.ssmrmh.ro}\end{matrix}\right) \overset{?}{\leq} \frac{(4R+r)}{s^2r}\cdot\frac{s^2\sqrt{s^2-16Rr+21r^2}}{4}$$

$$\Leftrightarrow (1296R^2+648Rr-6975r^2)s^6$$
$$-r(19008R^3-18000R^2r-42132Rr^2-11463r^3)s^4$$
$$-r^2(27072R^4-18336R^3r-47700R^2r^2-17824Rr^3-1867r^4)s^2-$$
$$r^3(9216R^5-4416R^4r-11808R^3r^2-5604R^2r^3-1044Rr^4-69r^5)\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 0 \text{ and } \because$$

$$P = (1296R^2+648Rr-6480r^2)(s^2-16Rr+5r^2)^3-$$
$$495r^2s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3) \overset{\text{Gerretsen,Double}-\text{Rouche}}{\geq} 0$$

$$\therefore \text{ in order to prove } (*)\text{, it suffices to prove : LHS of } (*) \overset{?}{\geq} P \Leftrightarrow X =$$
$$\left(16(2700R^3+1730R^2r+18033Rr^2+6853r^3)+4r(R^2+r^2)\right)s^4-$$
$$128r(7987R^4-1363R^3r-41109R^2r^2+24494Rr^3-3816r^4)s^2+$$
$$2r(32R^4+16R^3r+6R^2r^2+2Rr^3+43r^4)s^2+$$
$$256r^2(20700R^5-9055R^4r-107279R^3r^2+99626R^2r^3-30688Rr^4+3164r^5)$$
$$+r^3(64R^4+32R^3r+148R^2r^2+172Rr^3+85r^4)\ \boxed{\overset{?}{\underset{(**)}{\geq}}}\ 0$$

$\boxed{\textit{Case 1}}$ $X \geq 0$ *and* then, since $X(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to

$$\text{prove } (**)\text{, it suffices to prove : LHS of } (**) \overset{?}{\geq} X(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow U = 32(2812R^4+4908R^3r-33186R^2r^2+25460Rr^3-4752r^4)s^2+$$
$$(16R^4+24R^3r+3R^2r^2+15Rr^3+22r^4)s^2\ \boxed{\overset{?}{\underset{①}{\geq}}}$$
$$256r(5625R^5-2435R^4r+48583R^3r^2-48265R^2r^3+16505Rr^4-1886r^5)+$$
$$r^2(80R^4+56R^3r+4R^2r^2+151Rr^3+2r^4)$$

$\boxed{\textit{Case 1a}}$ $U \geq 0$ *and* then, LHS of ① $\overset{\text{Gerretsen}}{\geq} U(16Rr-5r^2) \overset{?}{\geq}$ RHS of ① $\Leftrightarrow$

$$(t-2)(128(11250t^3-19214t^2+8368t-1084)+112t^2+5t+54) \overset{?}{\geq} 0$$

$$\left(t=\frac{R}{r}\right)\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (1) \text{ is true}$$

$\boxed{\text{Case 1b}}$ $U<0$ and then, LHS of $(1) \overset{\text{Gerretsen}}{\geq} U(4R^2+4Rr+3r^2) \overset{?}{\geq}$ RHS of (1)

$$\Leftrightarrow (t-2)\begin{pmatrix}128\left((t-2)\begin{pmatrix}2812t^4+7721t^3-11403t^2+\\16629t+11414\end{pmatrix}+26496\right)+\\(t-2)(64t^4+32t^3+108t^2-8t+17)\end{pmatrix}\overset{?}{\geq} 0$$

$\to$ true $\because t \overset{\text{Euler}}{\geq} 2$ and combining cases $(1a)$ and $(1b)$, (1) is true $\forall\, \Delta ABC$

$\boxed{\text{Case 2}}$ $X<0$ and then, since $Q = X(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)+$

$$16\begin{pmatrix}10800R^5-2979R^4r-32025R^3r^2-\\7836R^2r^3-22820Rr^4+16816r^5\end{pmatrix}(s^2-4R^2-4Rr-3r^2) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Double-Rouche}}}{\geq} 0$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \overset{?}{\geq} Q \Leftrightarrow$

$$(t-2)\begin{pmatrix}64\left((t-2)\begin{pmatrix}675t^5+488t^4-1007t^3+\\5353t^2+1507t+6992\end{pmatrix}+13248\right)\\+(t-2)(52t^4+32t^3+11t^2-24t+4)\end{pmatrix}\overset{?}{\geq} 0 \to \text{true}$$

$\Rightarrow (**)$ is true and combining cases (1) and (2), $(**) \Rightarrow (*)$ is true $\forall\, \Delta ABC$

$$\therefore \frac{p_a+p_b+p_c}{r_a+r_b+r_c} \leq \frac{s^2\sqrt{s^2-16Rr+21r^2}}{4r_ar_br_c} \overset{\text{Dang Ngoc Minh}}{\leq} \frac{m_am_bm_c}{r_ar_br_c}$$

(Reference $\to$ Inequality in Triangle by Dang Ngoc Minh – 61; www.ssmrmh.ro)

$$\left(\sum_{cyc} p_a\right)^2 \cdot \prod_{cyc} g_a^2 \overset{\text{Ben Ajiba}}{\geq} \frac{100s^2-9r(64R-53r)}{25}\cdot\frac{r^4\begin{pmatrix}(9R+9r)s^2+\\(4R+r)^3\end{pmatrix}}{R} \overset{?}{\geq} \frac{4(4R+r)^2r^4s^4}{R^2}$$

(Reference; Inequality in Triangle by Mohamed Amine Ben Ajiba 65; www.ssmrmh.ro)

$$\Leftrightarrow -(700R^2-100Rr+100r^2)s^4+(6400R^4-384R^3r+309R^2r^2+4393Rr^3)s^2$$

$$-Rr(36864R^4-2880R^3r-15984R^2r^2-5148Rr^3-477r^4) \underbrace{\overset{?}{\geq}}_{(\bullet)} 0 \text{ and } \because T=$$

$$-(700R^2-100Rr+100r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$

$\overset{\text{Double-Rouche}}{\geq} 0 \therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \overset{?}{\geq} T \Leftrightarrow$

$$(W=3600R^4-13984R^3r+3309R^2r^2+2193Rr^3+200r^4)s^2+$$

$$r(7936R^5+30080R^4r+25984R^3r^2+9448R^2r^3+1577Rr^4+100r^5) \underbrace{\overset{?}{\geq}}_{(\bullet\bullet)} 0$$

$\to$ true for $W\geq 0$; for $W<0$, $\text{LHS}_{(\bullet\bullet)} \overset{\text{Gerretsen}}{\geq} W(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS}_{(\bullet\bullet)}$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}14400t^4+24000t^3+36580t^2+\\56360t+108067\end{pmatrix}+215784\right)\overset{?}{\geq} 0 \to \text{true}$$

$$\Rightarrow (\bullet\bullet),(\bullet) \text{ is true} \therefore \frac{m_am_bm_c}{r_ar_br_c} \geq \frac{\sum_{cyc} p_a}{\sum_{cyc} r_a} \geq \frac{h_ah_bh_c}{g_ag_bg_c}\ \forall\, ABC, \text{"="} \text{ iff } a=b=c \text{ (QED)}$$