ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{n_a+n_b+n_c}{p_a+p_b+p_c}\ge\frac{m_a+m_b+m_c}{\sqrt{3}s}\ge\frac{g_a+g_b+g_c}{m_a+m_b+m_c}$$

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$$\sum_{cyc}p_a\le\frac{\sqrt{21s^2+4Rr+r^2}}{9s^2+6Rr+r^2}\sqrt{\begin{matrix}(s^2-16Rr-3r^2)(21s^2+4Rr+r^2)+\\4(8s^2(8Rr+2r^2)-4s^2(2Rr+2r^2)+4Rrs^2)\end{matrix}}$$

$$\left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle by Dang Ngoc Minh}-129;\\\text{published at www.ssmrmh.ro}\end{matrix}\right),$$

$$\sum_{cyc}m_a\overset{\text{Chu-Yang}}{\le}\sqrt{4s^2-16Rr+5r^2}\text{ and }\sum_{cyc}n_a\overset{\text{Ben Ajiba}}{\ge}\sqrt{9s^2-80Rr-2r^2}$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba}-74;\\\text{published at www.ssmrmh.ro}\end{matrix}\right)$$

$$\therefore\text{ in order to prove : }\frac{n_a+n_b+n_c}{p_a+p_b+p_c}\ge\frac{m_a+m_b+m_c}{\sqrt{3}s},\text{it suffices to prove :}$$

$$3s^2(9s^2-80Rr-2r^2)\overset{?}{\ge}$$
$$\frac{(4s^2-16Rr+5r^2)(21s^2+4Rr+r^2)}{(9s^2+6Rr+r^2)^2}\cdot\left(\begin{matrix}(s^2-16Rr-3r^2)(21s^2+4Rr+r^2)+\\4(8s^2(8Rr+2r^2)-4s^2(2Rr+2r^2)+4Rrs^2)\end{matrix}\right)$$
$$\Leftrightarrow 423s^8-(2076Rr-231r^2)s^6-r^2(47668R^2+1948Rr-3336r^2)s^4-$$
$$r^3(35008R^3+6632R^2r-2360Rr^2-471r^3)s^2-$$
$$r^4(4096R^4+1536R^3r-240R^2r^2-152Rr^3-15r^4)\boxed{\underset{(*)}{\overset{?}{\ge}}}0\text{ and }\because P=$$

$$423(s^2-16Rr+5r^2)^4+(24996Rr-8229r^2)(s^2-16Rr+5r^2)^3+$$
$$r^2(16(31400R^2-22863Rr+3957r^2)+12R^2+8Rr+9r^2)(s^2-16Rr+5r^2)^2+$$
$$12r^3\left(\begin{matrix}16(19665R^3-25603R^2r+9856Rr^2-1183r^3)+\\8R^2r+5Rr^2+6r^3\end{matrix}\right)(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\ge}0\therefore\text{ in order to prove }(*),\text{it suffices to prove : LHS of }(*)\overset{?}{\ge}P$$
$$\Leftrightarrow(t-2)(32(16800t^3-14678t^2+4265t-413)+16t^2+3t+26)\overset{?}{\ge}0$$
$$\left(t=\frac{R}{r}\right)\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{n_a+n_b+n_c}{p_a+p_b+p_c}\ge\frac{m_a+m_b+m_c}{\sqrt{3}s}\text{ and again,}\sum_{cyc}g_a\le\sqrt{2R+5r}.\sqrt{\frac{s^2+4Rr+r^2}{2R}}$$

$$\left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle by}\\\text{Dang Ngoc Minh}-258;\text{published at www.ssmrmh.ro}\end{matrix}\right)\text{ and}$$

$$\left(\sum_{cyc} m_a\right)^2 \overset{\text{Chu-Yang}}{\geq} 4s^2 - 28Rr + 29r^2 \text{ and so, in order to prove :}$$

$$\frac{m_a + m_b + m_c}{\sqrt{3}s} \geq \frac{g_a + g_b + g_c}{m_a + m_b + m_c}, \text{it suffices to prove :}$$

$$(4s^2 - 28Rr + 29r^2)^2 \overset{?}{\geq} 3s^2.\frac{(2R + 5r)(s^2 + 4Rr + r^2)}{2R}$$

$$\Leftrightarrow (26R - 15r)s^4 - r(472R^2 - 398Rr + 15r^2)s^2 +$$

$$r^2(1568R^3 - 3248R^2r + 1682Rr^2) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and } \because Q =$$

$$(26R - 15r)(s^2 - 16Rr + 5r^2)^2 + 3r(120R^2 - 114Rr + 45r^2)(s^2 - 16Rr + 5r^2)$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} Q$$

$$\Leftrightarrow (t - 2)(112t^2 - 196t + 25) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true}$$

$$\therefore \frac{m_a + m_b + m_c}{\sqrt{3}s} \geq \frac{g_a + g_b + g_c}{m_a + m_b + m_c} \text{ and so,}$$

$$\frac{n_a + n_b + n_c}{p_a + p_b + p_c} \geq \frac{m_a + m_b + m_c}{\sqrt{3}s} \geq \frac{g_a + g_b + g_c}{m_a + m_b + m_c} \ \forall \ ABC,$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$